Positive Weights on the Efficient Frontier

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Acknowledgments

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Background

- Markowitz: Portfolio selection
- Mean variance efficiency
- Sharpe: Equilibrium model: CAPM
- Investor demands equal supply
- Implication: market portfolio is on efficient frontier
- Market portfolio has positive weights
Positive Weights

- When will frontier portfolio have positive weights?
- Problem of theoretical interest and practical importance
- Best and Grauer (1985), Green (1986)
- Brennan and Lo (2010) show that portfolios on efficient frontier involve short positions
- Dick Roll
  "The dark cloud hanging over one of the most fundamental models of modern finance"
- Levy and Roll (2010) use reverse engineering to challenge this assertion
Problem of estimating $\mu$

- Very hard to estimate expected return vector
- Optimal portfolios highly sensitive to this input
- Benefits of optimization swamped by estimation errors
- One approach ignore historical information $\frac{1}{n}$ method
- Other approach use equilibrium method Black Litterman
This paper

• Provides a simple method to obtain a frontier portfolio with positive weights
• Uses eigenvectors of correlation matrix to construct set of orthogonal portfolios
• If any one of these orthogonal portfolios is on the frontier the others have the same expected return
• The portfolio corresponding to the largest eigenvalue has positive weights
• Ensure this portfolio is on the frontier
• This leads to nice results and intuitive interpretations
• Simple and natural way to obtain $\mu$
Assumptions

- There are $n$ risky assets
- Covariance matrix $\mathbf{V}$ is positive definite
- The correlation matrix, $\mathbf{C}$ is given by

$$\mathbf{V} = \mathbf{SCS}$$

where

$$\mathbf{S} = \begin{bmatrix}
\sigma_1 & 0 & \cdots & 0 \\
0 & \sigma_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \sigma_n
\end{bmatrix}$$

and $\sigma_i$ is standard deviation of asset $i$. 
Eigenvectors of $\mathbf{C}$

The matrix $\mathbf{C}$ has $n$ positive eigenvalues

$$\lambda_1 > \lambda_2 > \cdots \lambda_n > 0.$$ 

and $n$ associated eigenvectors

$$\mathbf{v}^{(1)}, \mathbf{v}^{(2)} \ldots \mathbf{v}^{(n)}.$$ 

The eigenvectors are pairwise orthogonal in the linear algebra sense.

$$\left(\mathbf{v}^{(i)}\right)^T \mathbf{v}^{(j)} = 0, \quad i \neq j \quad (1)$$

The principal eigenvector $\mathbf{v}^{(1)}$ is associated with the largest eigenvalue $\lambda_1$. 
**First Step.** Divide each component by the volatility of the corresponding asset

\[ w^{(i)} = S^{-1} v^{(i)}, \quad i = 1, 2 \cdots n. \]

**Second Step** Normalize

\[ y^{(i)} = \frac{w^{(i)}}{e^T w^{(i)}} = \frac{w^{(i)}}{k_i}, \quad i = 1, 2 \cdots n. \]

where \( e \) is the \( n \) by one vector of ones and \( k_i = e^T w^{(i)} \).
Properties of the $y^{(i)}$ portfolios

These portfolios have three useful properties

1. They are pairwise orthogonal. For $i \neq j$

$$
\begin{aligned}
\left(y^{(i)}\right)^T V^{y(j)} &= 0
\end{aligned}
$$

2. The weighting on a given stock is inversely proportional to its volatility.

3. The weights in the portfolio $y^{(1)}$ are typically positive.
Assume portfolio $\mathbf{y}^{(1)}$ is on the frontier. It has expected return $\mu_m$. The other $(n - 1)$ orthogonal portfolios have the same expected return, $\mu_z$. This gives us $n$ conditions to determine $\mu$ the expected returns on the individual assets. We have

$$
\begin{align*}
\left( \mathbf{y}^{(1)} \right)^T \mu &= \mu_m \\
\left( \mathbf{y}^{(j)} \right)^T \mu &= \mu_z, \quad j = 2, 3 \cdots n
\end{align*}
$$

These give $\mu$ in terms of $\mu_m, \mu_z$ and the weights on the $n$ orthogonal portfolios.
Numerical Example, \( n = 3 \)

Assume

\[
V = \begin{bmatrix}
0.0256 & 0.0256 & 0.0164 \\
0.0256 & 0.0400 & 0.0400 \\
0.0164 & 0.0400 & 0.0625
\end{bmatrix}
\quad S = \begin{bmatrix}
0.16 & 0.00 & 0.00 \\
0.00 & 0.20 & 0.00 \\
0.00 & 0.00 & 0.25
\end{bmatrix}
\]

The three eigenvalues of the correlation matrix are

\[
\lambda_1 = 2.3546, \quad \lambda_2 = 0.5904, \quad \lambda_3 = 0.0550.
\]
The three orthogonal portfolios

The weights in the three orthogonal portfolios are as follows

\[ y^{(1)} = \begin{bmatrix} 0.3868 \\ 0.3656 \\ 0.2476 \end{bmatrix} \quad y^{(2)} = \begin{bmatrix} 2.7778 \\ 0.0000 \\ -1.7778 \end{bmatrix} \quad y^{(3)} = \begin{bmatrix} 3.5033 \\ -4.7455 \\ 2.2421 \end{bmatrix} \]

Assuming \( \mu_m = 0.12 \), \( \mu_z = 0 \), the expected return vector is

\[
\begin{bmatrix}
\mu_1 \\ \mu_2 \\ \mu_3 \\
\end{bmatrix} = \begin{bmatrix} 0.0914 \\ 0.1349 \\ 0.1428 \end{bmatrix}
\]
Plot of the three orthogonal portfolios

Portfolio $y_1$ is on the frontier.

Portfolios $y_2$ and $y_3$ lie on the zero beta line.

Figure: Efficient frontier and the three orthogonal factor portfolios.
Conditions for Positive Weights

A sufficient condition for the existence of a dominant eigenvector with positive entries is that all the pairwise correlations are positive. This is a consequence of the classic Perron-Frobenius theorem. Result has been extended to allow for some negative correlation.

Definition One

An \( n \times n \) matrix \( A \) is said to possess the strong Perron-Frobenius property if its dominant eigenvalue \( \lambda_1 \) is positive and each element of the corresponding eigenvector \( v^{(1)} \) is positive.

Definition Two

An \( n \times n \) matrix \( A \) is said to be eventually positive if there exists a positive integer \( k_0 \) such that \( A^k > 0 \) for all \( k > k_0 \).
Theorem (Noutsos)

For any symmetric $n \times n$ matrix $A$ the following properties are equivalent.

1. $A$ possesses the strong Frobenius-Perron property.
2. $A$ is eventually positive.

The intuition behind this result is straightforward. The matrices $A$ and $A^k$ have the same eigenvectors. If $A^k$ is positive then we know from the Perron Frobenius theorem that its dominant eigenvector has all its components positive. Hence the corresponding dominant eigenvector of $A$ has all its elements positive.
**Numerical Example**

Suppose $\mathbf{C}$ is given by

$$
\begin{bmatrix}
1.00 & 0.58 & 0.45 \\
0.58 & 1.00 & -0.35 \\
0.45 & -0.35 & 1.00
\end{bmatrix}
$$

The $(2,3)$ and $(3,2)$ components are negative. The eigenvalues are all positive so $\mathbf{C}$ is positive definite. The eigenvectors are

$$
\mathbf{v}^{(1)} = \begin{bmatrix}
0.7595 \\
0.6148 \\
0.2126
\end{bmatrix}, \quad
\mathbf{v}^{(2)} = \begin{bmatrix}
0.1966 \\
-0.5285 \\
0.8259
\end{bmatrix}, \quad
\mathbf{v}^{(3)} = \begin{bmatrix}
0.6201 \\
-0.5854 \\
-0.5223
\end{bmatrix},
$$

All the components of $\mathbf{v}^{(1)}$ are positive. The matrix $\mathbf{C}$ is eventually positive since $\mathbf{C}^k > 0$ for all $k \geq 7$. 
Practical solution

If there is too much negative correlation we can always get a dominant eigenvector with positive coefficients using a shrinkage estimate. Ledoit and Wolf (2004) show that using a covariance matrix that is a convex combination of the sample covariance matrix $\Sigma$ and a shrinkage target matrix, $F$ outperforms the stand alone sample matrix in terms of portfolio performance. The shrinkage estimator is

$$\delta F + (1 - \delta)\Sigma$$

There exists a $\delta_0 > 0$ such the shrinkage estimator has a dominant eigenvector with all positive weights for $\delta \geq \delta_0$. 
First we recall Green’s result.

*Theorem 1 of Green’s paper (1986)* states that a necessary and sufficient condition for a frontier portfolio to have strictly positive weights on all assets is that there must exist no nontrivial

(i) *hedge positions with expected payoffs equal to zero and non-negative correlation with all assets*

or

(ii) *portfolios that are either non-negatively or non-positively correlated with all assets and have expected returns equal to the zero-beta rate* \( \mu_Z \).
Suppose Green’s portfolio, \( \mathbf{h} \) exists. Note

\[
\left( y^{(1)} \right)^T \mathbf{Vh} = 0
\]  

(3)

The covariance between \( \mathbf{h} \) and the primitive assets is \( \mathbf{Vh} \). We have

\[
\left( y^{(1)} \right)^T \mathbf{Vh} > 0
\]

This contradicts equation (3).
Empirical application

We apply this method to the 30 stocks in the Dow Jones Industrial Average. Used weekly price data from June 14 2001 to February 27 2012 to estimate the correlation matrix. All the elements of the estimated correlation matrix are positive. Dominant eigenvector has all positive entries. All the other eigenvectors will have at least one negative sign.

We use a simple convention to summarize the elements of the 30 portfolios. For each portfolio, use $+1$ if a weight is positive and $-1$ if weight is negative. Then add these 30 numbers up.
Summary of eigenvectors based on 30 DJIA stocks.

<table>
<thead>
<tr>
<th>Eigen-vector</th>
<th>Sum of signs</th>
<th>Eigen-vector</th>
<th>Sum of signs</th>
<th>Eigen-vector</th>
<th>Sum of signs</th>
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<td>-4</td>
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<td>-2</td>
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<td>2</td>
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<td>19</td>
<td>4</td>
<td>29</td>
<td>4</td>
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<tr>
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<td>0</td>
<td>20</td>
<td>0</td>
<td>30</td>
<td>2</td>
</tr>
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</table>
Define gross leverage as

\[
\frac{\text{Sum of longs} + |\text{Sum of shorts}|}{\text{Net equity}} - 1
\]

We subtract one so that the long only portfolio has a leverage of zero. The other 29 DJIA portfolios are highly levered.
## Gross Leverage of the 30 DJIA portfolios

<table>
<thead>
<tr>
<th>Portfolio number</th>
<th>Gross leverage</th>
<th>Portfolio number</th>
<th>Gross leverage</th>
<th>Portfolio number</th>
<th>Gross leverage</th>
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<td>2</td>
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<td>22</td>
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<td>7</td>
<td>198.5</td>
<td>17</td>
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<td>27</td>
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<tr>
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<td>18</td>
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<td>9</td>
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<td>29</td>
<td>99.4</td>
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<tr>
<td>10</td>
<td>55.8</td>
<td>20</td>
<td>24.2</td>
<td>30</td>
<td>75.9</td>
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</tbody>
</table>
Results assuming $\mu_m = .08, \mu_z = 0$

<table>
<thead>
<tr>
<th>Ticker symbol</th>
<th>Weight in frontier portfolio %</th>
<th>Expected return on stock % pa</th>
<th>Standard deviation of stock % pa</th>
</tr>
</thead>
<tbody>
<tr>
<td>MMM</td>
<td>4.60</td>
<td>7.50</td>
<td>23.20</td>
</tr>
<tr>
<td>AA</td>
<td>2.42</td>
<td>14.45</td>
<td>44.37</td>
</tr>
<tr>
<td>AXP</td>
<td>2.89</td>
<td>12.69</td>
<td>38.04</td>
</tr>
<tr>
<td>T</td>
<td>3.28</td>
<td>6.38</td>
<td>25.35</td>
</tr>
<tr>
<td>BAC</td>
<td>1.67</td>
<td>15.33</td>
<td>55.09</td>
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<tr>
<td>BA</td>
<td>2.95</td>
<td>10.29</td>
<td>33.91</td>
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<tr>
<td>CAT</td>
<td>3.00</td>
<td>11.40</td>
<td>35.40</td>
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<tr>
<td>CVX</td>
<td>4.04</td>
<td>7.52</td>
<td>24.79</td>
</tr>
<tr>
<td>CSCO</td>
<td>2.44</td>
<td>9.57</td>
<td>35.96</td>
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<tr>
<td>KO</td>
<td>4.15</td>
<td>4.65</td>
<td>19.22</td>
</tr>
</tbody>
</table>
Robustness of results
Distribution of weights

Figure: Distribution of portfolio weights for DJ example.
Figure: Distribution of expected returns for DJ example.
**Figure:** Performance of our method (in red) with $\frac{1}{n}$ method (in blue) based on 30 DJ stocks
Summary

• Simple approach to find positive portfolio
• Uses just the covariance matrix
• Natural way to obtain the return vector
• Method is robust estimation error
• Performance comparable with $\frac{1}{n}$ results