Foreword

This booklet is typed based on professor Barry Coly's lecture notes for ABIZ 7940 Production Economics in Winter 2008. I am responsible for all the errors and typos.

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Chapter 1
Static Cost Minimization

Consider a firm producing a single output \( y \) using \( N \) inputs \( x = (x_1, \ldots, x_N) \) according to a production function \( y = f(x) \). The firm is a price taker in its \( N \) factor markets, i.e. the firm treats factor prices \( w = (w_1, \ldots, w_N) \) as given. We assume that all inputs are freely adjustable and perfect rental markets exist for all capital goods, i.e. for now we ignore all costs of adjustment that can lead to dynamic behavior.

A necessary condition for static profit maximization is static cost minimization, i.e. at the profit maximizing level of output \( y \) and input prices \( w \) the firm necessarily solves the following cost minimization problem:

\[
\min_{x \geq 0} \sum_{i=1}^{N} w_i x_i = w^* x
\]

s.t. \( f(x) \geq y \)  

(1.1)

The minimum cost \( c \equiv w^* x \) to problem (1.1) depends on the levels of input prices \( w \) and output \( y \), and of course on the production function \( y = f(x) \). By solving (1.1) using different value of \( (w, y) \) we can in principle trace out the relation between minimum cost \( c \) and parameters \( (w, y) \), conditional on the firm’s particular production function \( y = f(x) \). This relation \( c = c(w, y) \) between minimum cost and parameters \( (w, y) \) is called the firm’s dual cost function.

1.1 Properties of \( c(w, y) \)

Property 1.1.

a) \( c(w, y) \) is increasing (or, more precisely, non-decreasing) in all parameters \( (w, y) \).
Fig. 1.1 $c(w, y)$ is concave in $w$

\[
c(w_A, w_B, y)
\]

b) $c(w, y)$ is linear homogeneous in $w$. i.e. $c(\lambda w, y) = \lambda c(w, y)$ for all scaler $\lambda > 0$ (if all factor prices $w$ increase by the same proportion, e.g. 10%, then the minimum cost of attaining the same level of output $y$ also increases by this proportion)

c) $c(w, y)$ is concave in $w$. i.e. $c(\lambda w_A + (1 - \lambda)w_B, y) \geq \lambda c(w_A, y) + (1 - \lambda)c(w_B, y)$ for all $0 \leq \lambda \leq 1$. See Figure 1.1

d) If $c(w, y)$ is differentiable in $w$, then

\[
x_i(w, y) = \frac{\partial c(w, y)}{\partial w_i}
\]

(Shephard’s Lemma)\(^1\)

Proof. Property 1.1.a is obvious from Equation (1.1), and 1.1.b follows from the fact that an equiproportional change in all factor prices $w$ does not change relative factor prices and hence does not change the cost-minimizing level of inputs $x^*$ for problem (1.1). 1.1.c is not so obvious. In order to prove it simply note that, for $w_C = \lambda w_A + (1 - \lambda)w_B$ and $x_C^*$ solving (1.1) for $(w_C, y)$,

\[
c(w_C, y) = w_Cx_C^*
= \lambda w_Ax_C^* + (1 - \lambda)w_Bx_C^*
\geq \lambda c(w_A, y) + (1 - \lambda)c(w_B, y)
\]  

since $w_Ax_C^* \geq c(w_A, y)$ and $w_Bx_C^* \geq c(w_B, y)$ (e.g. $w_Ax_C^*$ cannot be less than the minimum cost for problem (1.1) given prices $w_A \neq w_C$; $w_Ax_C^* > c(w_A, y)$ unless $x_C^*$ solves (1.1) for prices $w_A$ as well as for prices $w_C$). Numerous proofs of Shephard’s Lemma 1.1.d are available. Here we simply consider the most obvious method of proof (see Varian 1992 for alternative methods).

Expressing (1.1) in Lagrange form

\(^1\) Note that $c(w, y)$ can be differentiable in $w$ even if, e.g. the production function $y = f(x)$ is Leontief (fixed proportions). In general differentiability of $c(w, y)$ is a weaker assumption than differentiability of $y$.\n
1.2 Corresponding properties of $x(w, y)$ solving problem (1.1)

$$
\min_{x, \lambda} w x - \lambda (f(x) - y) = w x^* - \lambda^* (f(x^*) - y) = c(w, y) \quad (1.1')
$$

with first order condition

$$
w_i - \lambda^* \frac{\partial f(x^*)}{\partial x_i} = 0
$$
$$
f(x^*) - y = 0
$$

Then $\frac{\partial c(w, y)}{\partial w}$ can be calculated by total differentiation as follows:

$$
\frac{\partial c(w, y)}{\partial w_i} = x_i^* + \sum_{j=1}^{N} \frac{\partial x_i^*}{\partial w_j} (w_j - \lambda^* \frac{\partial f(x^*)}{\partial x_j}) - \frac{\partial \lambda^*}{\partial w_i} (f(x^*) - y) \quad (1.3)
$$

by the first order conditions to problem (1.1'). □

It is important to note that Shephard's Lemma 1.1.d is simply an application of the envelope theorem (Samuelson 1947). The lemma states that, for an infinitesimal change in factor price $w_i$ (all other factor prices and output remaining constant), the change in minimum cost divided by the change in $w_i$ is equal to the equilibrium level of input $i$ in the absence of any change in $(w, y)$. In other words, in the limit, zero changes in equilibrium $x^*$ in response to a change in $w_i$ are optimal. Obviously such a lemma has no economic content, i.e. does not describe optimal response to finite changes in $w_i$. Nevertheless Shephard's and analogous envelope theorems are critical to the empirical and theoretical application of duality theory. This distinction is easily missed in more complex models.

1.2 Corresponding properties of $x(w, y)$ solving problem (1.1)

**Property 1.2.**

a) $x(w, y)$ is homogeneous of degree 0 in $w$. i.e. $x(\lambda w, y) = x(w, y)$ for all scalar $\lambda \geq 0$.

b) $\left[ \frac{\partial^2 c(w, y)}{\partial w \partial w} \right]_{N \times N}$ is symmetric negative semidefinite.

**Proof.** 1.2.a simply states that the cost minimizing solution $x^*$ to problem (1.1) depends only on relative prices. In order to prove 1.2.b, note that the Hessian matrix $\left[ \frac{\partial^2 c(w, y)}{\partial w \partial w} \right]_{N \times N}$ is symmetric negative semidefinite by concavity and twice differentiability of $c(w, y)$ in $w$, and then note that $\frac{\partial c(w, y)}{\partial w} = x(w, y)$ for all $w$ (Shephard's Lemma) □

[...content continues...]
In order to test economics theories it is important to know all of the restrictions that are placed on observable behavior by particular theories. This is known as the integrability problem in economics. It can easily be shown that 1.2.a-b exhausted the (local) properties that are placed on factor demands \( x(w, y) \) by the hypothesis of cost minimization (1.1).

**Proof.** It has already been shown that the properties 1.1 of the cost function imply 1.2. In order to show that 1.2 exhausts the implications of cost minimization (1.1) for local properties of \( x(w, y) \), first total differentiate \( c(w, y) = wx(w, y) \) with respect to \( w_i \),

\[
\frac{\partial c(w, y)}{\partial w_i} \equiv x_i(w, y) + \sum_{j=1}^{N} w_j \frac{\partial x_j(w, y)}{\partial w_i} \quad i = 1, \ldots, N. \tag{1.4}
\]

1.2.a implies (by Euler’s theorem\(^2\)) \( \sum_{j=1}^{N} w_j \frac{\partial x_j(w, y)}{\partial w_i} = 0 \) \( (i = 1, \ldots, N) \) and together with symmetry 1.2.b this reduces the identity (1.4) to \( \frac{\partial c(w, y)}{\partial w_i} = x_i(w, y) \geq 0 \) \( (i = 1, \ldots, N) \) (Shephard’s Lemma). In addition \( \frac{\partial^2 c(w, y)}{\partial w \partial w_i} \) \( (N \times N) \) 1.2.b implies that the system of differential equations \( x_i(w, y) = \frac{\partial c(w, y)}{\partial w_i} \) \( (i = 1, \ldots, N) \) integrates up to an underlying cost function \( c(w, y) \) (Frobenius theorem\(^3\)). Shephard’s Lemma also implies (by simple differentiation of \( x(w, y) = \frac{\partial c(w, y)}{\partial w} \) with respect to \( w \)) \( \frac{\partial^2 c(w, y)}{\partial w \partial w_i} \) \( (N \times N) \) negative semidefinite by 1.2.b. It can then be shown that \( \frac{\partial^2 c(w, y)}{\partial w \partial w_i} \) \( (N \times N) \) negative semidefinite implies \( y = f(x) \) is quasiconcave at \( x = x(w, y) \) (see (1.6) below). This establishes the second order conditions on the production function \( y = f(x) \) for competitive cost minimization. The first order condition follow from the fact that 1.2 establishes Shephard’s Lemma for all \( \frac{\partial x(w, y)}{\partial w}, \frac{\partial y(w, y)}{\partial w}, \) satisfying 1.2 (see (1.3)). \( \square \)

### 1.3 Second order relations between \( c(w, y) \) and \( f(x) \)

It is sometimes interesting to ask whether or not the firm’s production function \( y = f(x) \) can be recovered from knowledge of the firm’s cost function \( c(w, y) \), i.e. can we construct \( y = f(x) \) directly from knowledge of \( c(w, y) \)\? The answer is essentially yes (the only qualification is that we cannot recover \( f(x) \) at levels of \( x \) that cannot be solutions to a cost minimization problem (1.1) for some \( (w, y) \), i.e. at levels of associated with locally non-convex isoquants). For example, given knowl-

\(^2\) Euler’s theorem states that, if \( g(\lambda x) = \lambda^r g(x) \) for all scalar \( \lambda > 0 \) (i.e. the function \( g(x) \) is homogenous of degree \( r \) ), then \( rg(x) = \sum_{i=1}^{r} x_i \frac{\partial g(x)}{\partial x_i} \) and \( (r-1) \frac{\partial^2 g(x)}{\partial x_i \partial x_j} = \sum_{i=1}^{r} x_i \frac{\partial^2 g(x)}{\partial x^r \partial x_i} \).  

\(^3\) The Frobenius theorem states that a system of differential equations \( g_i(v) = \frac{\partial \psi(v)}{\partial v_i} \) \( (i = 1, \cdots, T) \) has a solution \( \psi(v) \) if and only if \( \frac{\partial g_i(v)}{\partial v_j} = \frac{\partial g_j(v)}{\partial v_i} \) \( (i, j = 1, \cdots, T) \).
1.3 Second order relations between \( c(w, y) \) and \( f(x) \)

edge of \( c(w, y) \) and differentiability of \( c(w, y) \), application of Shephard’s Lemma
\[
\frac{\partial c}{\partial w_i} = x_1, \ldots, \frac{\partial c}{\partial w_N} = x_N
\]

immediately gives the cost-minimizing levels of inputs \( x \) corresponding to \( (w, y) \) (this assumes a unique solution \( x \) to problem (1.1)). By varying \((w, y)\) we can map out \( y = f(x) \) from \( c(w, y) \) in this manner.

A related point that will be important later (in a lecture on functional forms) is that the first and second derivatives of \( f(x) \) at \( x(w, y) \) can be calculated directly from knowledge of the first and second derivatives of \( c(w, y) \). The first derivatives can be calculated simply as

\[
\frac{\partial f(x(w, y))}{\partial x_i} = \frac{w_i}{\partial c(w, y)/\partial y} \quad i = 1, \cdots, N
\]

using the first order conditions \( \frac{\partial f(x)}{\partial x_i} - \lambda w_i = 0 \) (\( i = 1, \cdots, N \)) for cost minimization (1.1’), where \( \lambda \equiv \frac{\partial c(w, y)}{\partial y} \). The procedure for calculating the second derivatives of \( f(x) \) from \( c(w, y) \) is not quite as obvious. The corresponding formula in matrix notation is

\[
\begin{bmatrix}
\frac{\partial c(w, y)}{\partial y} & \frac{\partial^2 f}{\partial x \partial x} & \frac{\partial f}{\partial x} \\
\frac{\partial c(w, y)}{\partial x} & 0 & \mathbf{0}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial^2 c(w, y)}{\partial u \partial w} & \frac{\partial^2 c(w, y)}{\partial w \partial y} \\
\frac{\partial^2 c(w, y)}{\partial y \partial w} & \frac{\partial^2 c(w, y)}{\partial y \partial y}
\end{bmatrix}^{-1}
\]

where it is assumed without loss of generality that the above inverse exists

\[
\text{Proof.} \text{ Consider the case } N = 2 \text{ so } y = f(x_1, x_2) \text{ and } c = c(w_1, w_2, y). \text{ The first order condition for cost minimization (1.1’)} \text{ can be written as}
\]

\[
\begin{align*}
w_1 - c_y(w, y) f_{x_1} &= 0 \\
w_2 - c_y(w, y) f_{x_2} &= 0 \\
y &= f(x)
\end{align*}
\]

where for now \( c_y(w, y) \equiv \frac{\partial c(w, y)}{\partial y}, f_{x_1} \equiv \frac{\partial f(x)}{\partial x_1}, \text{ etc.} \)

Total differentiating these conditions (1.7) with respect to \((w_1, w_2, y)\) yields (using Shephard’s Lemma):

\[
\text{Proof.} \text{ Consider the case } N = 2 \text{ so } y = f(x_1, x_2) \text{ and } c = c(w_1, w_2, y). \text{ The first order condition for cost minimization (1.1’)} \text{ can be written as}
\]

\[
\begin{align*}
w_1 - c_y(w, y) f_{x_1} &= 0 \\
w_2 - c_y(w, y) f_{x_2} &= 0 \\
y &= f(x)
\end{align*}
\]

where for now \( c_y(w, y) \equiv \frac{\partial c(w, y)}{\partial y}, f_{x_1} \equiv \frac{\partial f(x)}{\partial x_1}, \text{ etc.} \)

Total differentiating these conditions (1.7) with respect to \((w_1, w_2, y)\) yields (using Shephard’s Lemma):
1.4 Additional properties of $c(w, y)$

Property 1.3.

a) $y = f(x)$ homothetic $\iff c(w, y) = \phi(y) c(w, 1)$ for some function $\phi$.

b) $y = f(x)$ constant returns to scale $\iff c(w, y) = y c(w, 1)$.

c) All the partial elasticity of substitution between inputs $i$ and $j$ (output $y$ constant)

$$
\sigma_{ij}(w, y) = \frac{\frac{\partial}{\partial w_i} f(x, y) / f(x, y)}{\frac{\partial}{\partial w_j} f(x, y) / f(x, y)}
$$

can be calculated simply as

$$
\sigma_{ij}(w, y) = \frac{c(w, y) \frac{\partial^2 c(w, y)}{\partial w_i \partial w_j}}{\frac{\partial c(w, y)}{\partial w_i} \frac{\partial c(w, y)}{\partial w_j}}
$$

(see Uzawa 1962, p. 291-9).

d) Assuming a vector of outputs $y = y_1, \ldots, y_M$. The transformation function $f(x, y) = 0$ is disjoint (i.e. $y_1 = f_1(x_1), \ldots, y_M = f_M(x_M)$ where input vector $x_1, \ldots, x_M$ do not overlap) only if $\frac{\partial^2 c(w, y)}{\partial y_i \partial y_j} = 0$ for all $i \neq j$, all $(w, y)$.
1.5 Applications of dual cost function in econometrics

The above theory is usually applied by first specifying a functional form \( \phi(w, y) \) for the cost function \( c(w, y) \) and differentiating \( \phi(w, y) \) with respect to \( w \) in order to obtain the estimating equations

\[
x_i = \frac{\partial \phi(w, y)}{\partial w_i} \quad i = 1, \ldots, N
\]  
(1.10)

(employing Shephard’s Lemma). Then the symmetry restrictions \( \frac{\partial^2 \phi}{\partial w_i \partial w_j} = \frac{\partial^2 \phi}{\partial w_j \partial w_i} \) \((i = 1, \ldots, N)\) are tested and the second order condition \( \left[ \frac{\partial^2 \phi}{\partial w \partial w} \right]_{N \times N} \) negative semidefinite is checked at all data points \((w, y)\).

For example a cost function could be postulated as having the functional form

\[
c = y \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} w_i^{\frac{1}{2}} w_j^{\frac{1}{2}}
\]

(a Generalized Leontief functional form with \( y = f(x) \) showing constant returns to scale), which leads to the following equations for estimation:

\[
x_i = \frac{\sum_{j=1}^{N} a_{ij} \left( \frac{w_j}{w_i} \right)^{\frac{1}{2}}}{y} \quad i = 1, \ldots, N
\]  
(1.11)

Here the symmetry restriction \( \frac{\partial x_i}{\partial w_j} = \frac{\partial x_j}{\partial w_i} \) are expressed as \( a_{ij} = a_{ji} \) \((i = 1, \ldots, N)\) which are easily tested. Equation (1.10) can be interpreted as being derived from a cost function \( c(w, y) \) for a producer showing static, competitive cost-minimizing behavior if and only if the symmetry and second order conditions are satisfied.

The major advantage of this approach is that it permits the specification of a system of factor demand equations \( x = x(w, y) \) that are consistent with cost minimization and with a very general specification of technology. In contrast, suppose that we wished to specify explicitly a solution to a cost minimization problem. Then we would estimate a production function directly with first order conditions for cost minimization:
\[ y = f(x) \]
\[ \frac{\partial f}{\partial x_i} = w_i \]
\[ \frac{\partial f}{\partial x_j} = w_j \]
\[ \frac{\partial f}{\partial x_k} = w_k \]

(1.12)

However, unless a very restrictive functional form is specified for the production function (e.g. Cobb-Douglas), then we can seldom drive the factor demand equations \( x = x(w, y) \) explicitly from (1.12). Since policy makers are usually more interested in demand and supply behavior than in production functions per se, this greater ease in specification of \( x(w, y) \) is an important advantage of duality theory.

Two other advantages of a duality approach rather than a primal approach (1.12) to the estimation of producer behavior are apparent. First, the hypothesis of competitive cost minimization is more readily tested in the framework of equations (1.10) than (1.12). Second, variables that are omitted from the econometric model (but are observed by producers) influence both the error terms and production decisions but do not necessarily influence factor prices to the same degree (e.g. factor prices may be exogenous to the industry). This tends to introduce greater simultaneous equations biases into the estimation of (1.12) than of (1.10).

One disadvantage of (1.10) is that output \( y \), as well as factor prices \( w \), is treated as exogenous. Since the firm generally in effect chooses \( y \) jointly with \( x \), this mis-specification can lead to simultaneous equations biases in the estimators. So the extent that production is constant return to scale with a single output, this difficulty can be avoided by using 1.3.b to specify a unit cost function \( c(w) = c(w, y)/y \) and applying Shephard’s Lemma to obtain estimating equations

\[ \frac{x_i}{y} = \frac{\partial c(w)}{\partial w_i} \quad i = 1, \ldots, N \]

(1.13)

for a given functional form \( c(w) \) (e.g. (1.11)). Under constant returns to scale \( x_i(w, y)/y \) depends on \( w \) but not on \( y \), so that (1.13) is well defined and estimation is independent of whether \( y \) is endogenous or exogenous to the firm.

A second disadvantage of this duality approach to the specification of functional forms for econometric models, and a disadvantage of primal approaches such as (1.12) as well, is that it is derived from the theory of the individual firm but is usually applied to market data that is aggregated over firms. Difficulties raised by such aggregation will be discussed in a later lecture.

1.6 Conclusion

The dual cost function approach offers many advantages in the estimation of production technologies. The estimated factor demands \( x = x(w, y) \) measure factor
substitution along an isoquant and the effects of scale of output on factor demands, and first and second derivatives of the production function can be calculated. Moreover the assumption of cost minimization is consistent with various broader theories of producer behavior.

However effective policy making depends more on a knowledge of producer behavior than of production functions per se. By ignoring the effect of output prices on the firm’s input levels, the dual cost function approach generally is inappropriate for the modeling of economic behavior.

Of course cost functions can be embedded within a broader behavioral model. For example, static competitive profit maximization implies a cost minimization model such as 1.1 together with first order conditions

\[ \frac{\partial c(w, y)}{\partial y} = p \]  

for optimal output levels (marginal cost equals output price). This equation implicitly defines the optimal level of \( y \) as \( y = y(w, p) \) provided of course that \( y \) enters (1.14), i.e. \( \frac{\partial c(w, y)}{\partial y} \) is not independent of \( y \) or equivalently \( f(x) \) does not show constant returns to scale. In this case equations 1.1.d, (1.10) and (1.14) can be estimated jointly, and the second order condition for profit maximization are expressed as \( c(w, y) \) concave and \( \frac{\partial^2 c(w, y)}{\partial y \partial p} \leq 0 \).

Nevertheless there can be substantial disadvantages to this approach to modeling competitive profit maximization. The assumption of constant returns to scale is commonly employed in empirical studies, and the assumption of profit maximization is not so easily tested here (the homogeneity and reciprocity condition for cost minimization do not imply integration up to a profit function). Therefore, for policy purposes, it is often better to model and test directly (using dual profit functions) the hypothesis of competitive profit maximization behavior.\(^5\)

References


\(^5\) In passing, note that this indirect approach to the modeling of profit maximization (1.1.d, (1.10), (1.14)) may be superior to a direct approach (see next lecture) when there is substantially higher multicollinearity between factor prices \( w \) and output prices \( p \) than between \( w \) and output levels \( y \).
Chapter 2
Static Competitive Profit Maximization

As before, consider a firm producing a single output \( y \) using \( N \) inputs \( x = (x_1, \cdots, x_N) \) according to a production function \( y = f(x) \), and assume that the firm is a price taker in all \( N \) factor markets. In addition, we now assume that the firm takes output price \( p \) as given and chooses its levels of inputs to solve the following static competitive profit maximization problem:

\[
\max_{x \geq 0} \left\{ pf(x) - \sum_{i=1}^{N} w_i x_i \right\} = pf(x^*) - wx^*.
\] (2.1)

The maximum profit \( \pi \equiv pf(x^*) - wx^* \) to problem (2.1) depends on prices \((w, p)\) and the firm’s production function \( y = f(x) \). The corresponding relation \( \pi = \pi(w, p) \) between maximum profit and prices is denoted as the firm’s dual profit function.

2.1 Properties of \( \pi(w, p) \)

Property 2.1.

a) \( \pi(w, p) \) is decreasing in \( w \) and increasing in \( p \).

b) \( \pi(w, p) \) is linear homogeneous in \((w, p)\). i.e., \( \pi(\lambda w, \lambda p) = \lambda \pi(w, p) \) for all scalar \( \lambda > 0 \).

c) \( \pi(w, p) \) is convex in \((w, p)\). i.e.,

\[
\pi [\lambda w_A + (1 - \lambda)w_B, \lambda p_A + (1 - \lambda) p_B] \leq \lambda \pi(w_A, p_A) + (1-\lambda)\pi(w_B, p_B)
\]

for all \( 0 \leq \lambda \leq 1 \). See Figure 6.1.
Fig. 2.1 \( \pi(w, p) \) is convex in \((w, p)\)

\[
\pi(w, p)
\]

\[\begin{array}{c}
\text{Proof.}\end{array}\]

Properties 2.1.a–b follow obviously from the definition of the firm’s maximization problem (2.1). In order to prove 2.1.c simply note that, for \((w_C, p_C) \equiv \lambda(w_A, p_A) + (1 - \lambda)(w_B, p_B)\) and \(x_C^*\) solving 2.1 for \((w_C, p_C)\),

\[
\pi(w_C, p_C) = p_C f(x_C^*) - w_C x_C^*
\]

\[
= \lambda [p_A f(x_C^*) - w_A x_C^*] + (1 - \lambda) [p_B f(x_C^*) - w_B x_C^*]
\]

\[
\leq \lambda \pi(w_A, p_A) + (1 - \lambda) \pi(w_B, p_B)
\]

since \(\pi(w_A, p_A) \geq p_A f(x_C^*) - w_A x_C^*\), \(\pi(w_B, p_B) \geq p_B f(x_C^*) - w_B x_C^*\). In order to prove Hotelling’s Lemma 2.1.d simply total differentiate \(\pi(w, p) \equiv p f(x^*) - w x^*\) with respect to \((w, p)\) respectively and then apply the standard first order conditions for an interior competitive profit maximum:

\[
\frac{\partial \pi(w, p)}{\partial p} = f(x^*) + \sum_{k=1}^{N} \left[ p \frac{\partial f(x^*)}{\partial x_k} - w_k \right] \frac{\partial x_k^*}{\partial p}
\]

\[
= f(x^*)
\]

\[
\frac{\partial \pi(w, p)}{\partial w_i} = -x_i^* + \sum_{k=1}^{N} \left[ p \frac{\partial f(x^*)}{\partial x_k} - w_k \right] \frac{\partial x_k^*}{\partial w_i}
\]

\[
= -x_i^* \quad i = 1, \ldots, N.
\]
2.2 Corresponding properties of $y(w, p)$ and $x(w, p)$

Hotelling’s Lemma plays the same role in the theory of competitive profit maximization as Shephard’s Lemma plays in the theory of competitive cost minimization. Hotelling’s Lemma is an envelope theorem. The Lemma applies only for infinitesimal changes in a price and yet is critical to the empirical theoretical application of dual profit functions.

### Property 2.2.

a) $y(w, p)$ and $x(w, p)$ are homogeneous of degree 0 in $(w, p)$. i.e.,
\[ y(\lambda w, \lambda p) = y(w, p) \] and \[ x(\lambda w, \lambda p) = x(w, p) \] for all scalar $\lambda \geq 0.$

b) \[
\begin{pmatrix}
\frac{\partial y(w, p)}{\partial w} & \frac{\partial y(w, p)}{\partial p} \\
\frac{\partial x(w, p)}{\partial w} & \frac{\partial x(w, p)}{\partial p}
\end{pmatrix}
\] is symmetric positive semidefinite.

**Proof.** Property 2.2.a follows directly from the maximization problem (2.1). In order to prove 2.2.b, note that
\[
\begin{pmatrix}
\frac{\partial^2 \pi(w, p)}{\partial w \partial p} \\
\frac{\partial^2 \pi(w, p)}{\partial w \partial p}
\end{pmatrix}
\] is symmetric positive semidefinite by 2.1.c and then apply 2.1.d to evaluate this matrix. □

Moreover, 2.2.a–b exhaust the (local) properties that are placed on output supply $y(w, p)$ and factor demand $x(w, p)$ relations by the hypothesis of competitive profit maximization (2.1).

**Proof.** First total differentiate $\pi(w, p) \equiv pf(x(w, p)) - w x(w, p)$ to obtain

\[
\frac{\partial \pi(w, p)}{\partial p} \equiv y(w, p) + p \frac{\partial y(w, p)}{\partial p} - \sum_{k=1}^{N} w_k \frac{\partial x_k(w, p)}{\partial p} \]

\[
\frac{\partial \pi(w, p)}{\partial w_i} \equiv -x_i(w, p) + p \frac{\partial y(w, p)}{\partial w_i} - \sum_{k=1}^{N} w_k \frac{\partial x_k(w, p)}{\partial w_i} \quad i = 1, \cdots, N
\]

(2.4)

Property 2.2.a implies (by Euler’s theorem)

\[\frac{\partial \pi(w, p)}{\partial w_i} = -x_i(w, p) \text{ homogeneous of degree 0.}\]

---

1 Alternatively, $\pi(w, p)$ homogeneous of degree one in $(w, p)$ implies (by Euler’s theorem) $\frac{\partial \pi(w, p)}{\partial w_i}$ homogeneous of degree 0.
and property 2.2.b states the reciprocity relations

\[ \frac{\partial y(w, p)}{\partial w_k} = -\frac{\partial x_k(w, p)}{\partial p}, \]
\[ \frac{\partial x_i(w, p)}{\partial w_k} = \frac{\partial x_k(w, p)}{\partial w_i}, \quad i, k = 1, \cdots, N, \]

so properties 2.2.a–b jointly imply

\[ p \frac{\partial y(w, p)}{\partial p} - \sum_{k=1}^{N} w_k \frac{\partial x_k(w, p)}{\partial p} = 0 \]
\[ p \frac{\partial y(w, p)}{\partial w_i} - \sum_{k=1}^{N} w_k \frac{\partial x_k(w, p)}{\partial w_i} = 0, \quad i = 1, \cdots, N. \]

Substituting this into the identity (2.4) yields

\[ \frac{\partial \pi(w, p)}{\partial p} = y(w, p) \geq 0 \] (Hotelling’s Lemma)
\[ \frac{\partial \pi(w, p)}{\partial w_i} = -x_i(w, p) \leq 0, \quad i = 1, \cdots, N. \]

The reciprocity relations 2.2.b imply that the system of differential equations \( \frac{\partial \pi(w, p)}{\partial p} = y(w, p), \frac{\partial \pi(w, p)}{\partial w_i} = -x_i(w, p), (i = 1, \cdots, N) \) integrates up to an underlying function \( \pi(w, p) \) (Frobenius theorem). The positive semidefiniteness restriction 2.2.b implies positive semidefiniteness of the Hessian matrix of \( \pi(w, p) \), and this in turn implies \( y = f(x) \) is concave at all \( x^* \) (see (2.6) below). This establishes the second order conditions on the production \( f(x) \) for competitive profit maximization. The first order conditions follow from the fact that 2.2 establish Hotelling’s Lemma for all \( \frac{\partial y(w, p)}{\partial p}, \frac{\partial x(w, p)}{\partial p}, \frac{\partial y(w, p)}{\partial w}, \frac{\partial x(w, p)}{\partial w} \) satisfying properties 2.2 (see (2.3)).

**2.3 Second order relations between \( \pi(w, p) \) and \( f(x), c(w, y) \)**

As in the case of a cost function, the firm’s production function \( y = f(x) \) can be recovered from knowledge of the profit function \( \pi(w, p) \). Given knowledge of
2.3 Second order relations between $\pi(w, p)$ and $f(x)$, $c(w, y)$

$\pi(w, p)$ and differentiability of $\pi(w, p)$, application of Hotelling’s Lemma immediately gives a profit maximizing combination $(x, y)$ for prices $(w, p)$.

Likewise the first and second derivatives of $f(x)$ at $x = x(w, p)$ can be calculated directly from the first and second derivatives of $\pi(w, p)$. The first derivatives can be calculated simply as

$$\frac{\partial f(x(w, p))}{\partial x_i} = \frac{w_i}{p} \quad i = 1, \ldots, N \quad (2.5)$$

using the first order conditions for an interior solution to problem (2.1). The second derivatives can be calculated from the matrix equation

$$p \otimes \left[ \frac{\partial^2 f(x(w, p))}{\partial x \partial x} \right]_{N \times N} = - \left[ \frac{\partial^2 \pi(w, p)}{\partial w \partial w} \right]_{N \times N}^{-1} \quad (2.6)$$

assuming an inverse for $\left[ \frac{\partial^2 \pi(w, p)}{\partial w \partial w} \right]_{N \times N}$.

**Proof.** Simply total differentiate the first order conditions $p \frac{\partial f(x^*)}{\partial x_i} - w_i = 0 (i = 1, \ldots, N)$ with respect to $w$ to obtain

$$p \sum_{k=1}^{N} \frac{\partial^2 f(x^*)}{\partial x_i \partial x_k} \frac{\partial x_k(w, p)}{\partial w_j} - 1 = 0 \quad i, j = 1, \ldots, N, \quad (2.7)$$

Substitute $\frac{\partial x_k(w, p)}{\partial w_j} = \frac{\partial^2 \pi(w, p)}{\partial w_i \partial w_j}$ (by Hotelling’s Lemma) into (2.7) and express the result in matrix form. \hfill \Box

This result (2.6) can easily be extended to the case of multiple outputs (Lau 1976).

Since elasticities of substitution (holding output $y$ constant) and scale effects are easily expressed in terms of a dual cost function $c(w, y)$, it is useful to note that $\pi(w, p)$ also provides a second order approximation to $c(w, y)$. The first derivatives of $c(w, y)$ can be calculated simply as

$$\frac{\partial c(w, y^*)}{\partial w_i} = x_i(w, y^*) = - \frac{\partial \pi(w, p)}{\partial w_i} \quad i = 1, \ldots, N \quad (2.8)$$

where $y^* \equiv y(w, p) = \frac{\partial \pi(w, p)}{\partial p}$, the profit maximizing level of output given $(w, p)$. The second derivatives of $c(w, y)$ at $y^* = y(w, p)$ can be calculated from $\pi(w, p)$ using the following matrix relations, here $c_{ww}(w, y) \equiv \left[ \frac{\partial^2 c(w, y)}{\partial w \partial w} \right]_{N \times N}$, $c_{wy}(w, y) \equiv \left[ \frac{\partial^2 c(w, y)}{\partial w \partial y} \right]_{N \times N}$, $\pi_{ww}(w, p) \equiv \left[ \frac{\partial^2 \pi(w, p)}{\partial w \partial w} \right]_{N \times N}$, $\pi_{wp}(w, p) \equiv \left[ \frac{\partial^2 \pi(w, p)}{\partial w \partial p} \right]_{N \times N}$.
\[
\left[ \frac{\partial^2 \pi(w, p)}{\partial w \partial p} \right]_{N \times N}
\]

\[
c_{ww}(w, y^*) = -\pi_{ww}(w, p) + \pi_{wp}(w, p) \pi_{pp}(w, p)^{-1} \pi_{wp}(w, p)^T
\]
\[
c_{wy}(w, y^*) = -\pi_{wp}(w, p) \pi_{pp}(w, p)^{-1}
\]
\[
c_{yy}(w, y^*) = \pi_{pp}(w, p)^{-1}
\] (2.9)

**Proof.** \(x_i(w, p) = x_i[w, y(w, p)]\) \((i = 1, \ldots, N)\) implies (by Hotelling’s Lemma and Shephard’s Lemma)

\[
-\pi_{w_i}(w, p) = c_{w_i}(w, y^*)
\] (2.10)

where \(y^* = y(w, p)\). Differentiating with respect to \((w, p)\),

\[
-\pi_{ww}(w, p) = c_{ww}(w, y^*) + c_{wy}(w, y^*) \pi_{wp}(w, p)^T
\]
\[
-\pi_{wp}(w, p) = c_{wy}(w, y^*) \pi_{pp}(w, p)
\] (2.11)

Combining (2.11),

\[
c_{ww}(w, y^*) = -\pi_{ww}(w, p) + \pi_{wp}(w, p) \pi_{pp}(w, p)^{-1} \pi_{wp}(w, p)^T
\]
\[
c_{wy}(w, y^*) = -\pi_{wp}(w, p) \pi_{pp}(w, p)^{-1}
\] (2.12)

Finally differentiating the first order condition \(c_y(w, y^*) = p\) (for profit maximization) with respect to \(p\) yields \(c_{yy}(w, y^*) = \pi_{pp}(w, p)^{-1}\).

### 2.4 Additional properties of \(\pi(w, p)\)

**Property 2.3.**

a) the partial elasticity of substitution \(\theta_{ij}\) between inputs \(i\) and \(j\), allowing for variation in output, can be defined as

\[
\theta_{ij}(w, p) = \frac{\pi(w, p)}{\pi_{w_i} \pi_{w_j}} \frac{\partial^2 \pi(w, p)}{\partial w_i \partial w_j}
\]

and, in the case of multiple outputs \(y = (y_1, \ldots, y_M)\), the partial elasticity of transformation between outputs \(i\) and \(j\) can be defined as

\[
t_{ij}(w, p) = \frac{\pi(w, p)}{\pi_{p_i} \pi_{p_j}} \frac{\partial^2 \pi(w, p)}{\partial p_i \partial p_j}
\]

\(i, j = 1, \ldots, N\)
2.5 Le Chatelier principles and restricted profit functions

Samuelson proved the following Le Chatelier principle: fixing an input at its initial static equilibrium level dampens own-price comparative static responses, or more precisely

\[
\frac{\partial^2 \pi(w, p)}{\partial p_i \partial p_j} = 0 \quad \text{for all } i \neq j, \text{ all } (w, p).
\]

where \( \bar{x}_1 \equiv x_1(w, p) \), i.e., \( \frac{\partial y(w, p)}{\partial p} \), \( \frac{\partial x(w, p)}{\partial w} \) denote the comparative static changes in \((y, x)\) when the level of input 1 cannot vary from its initial equilibrium level \( x_1(w, p) \) (Samuelson 1947). The following generalization of this result is easily established using duality theory:

\[
\begin{bmatrix}
\frac{\partial x(w, p)}{\partial p} & \frac{\partial y(w, p)}{\partial p} \\
\frac{\partial x(w, p)}{\partial w} & \frac{\partial y(w, p)}{\partial w}
\end{bmatrix}
= \begin{bmatrix}
\frac{\partial x(w, p, \bar{x}_1)}{\partial p} & \frac{\partial y(w, p, \bar{x}_1)}{\partial p} \\
\frac{\partial x(w, p, \bar{x}_1)}{\partial w} & \frac{\partial y(w, p, \bar{x}_1)}{\partial w}
\end{bmatrix}
\]

is a positive semidefinite matrix.

**Proof.** By the definition of competitive profit maximization (2.1),

\[
\pi(w, p) = \max_{x \geq 0} \left\{ p f(x) - \sum_{i=1}^{N} w_i x_i \right\} \geq \max_{x \geq 0} \left\{ p f(x) - \sum_{i=1}^{N} w_i x_i \right\} \equiv \pi(w, p, x_1^A)
\]

s.t. \( x_1 = x_1^A \) \hspace{1cm} (2.15)

i.e., adding a constraint \( x_1 = x_1^A \) to a maximization problem (2.1) generally decreases (and never increases) the maximum attainable profits. Then

\[
\phi(w, p, x_1^A) \equiv \pi(w, p) - \pi(w, p, x_1^A) \begin{cases} 
\geq 0 & \text{for all } (w, p, x_1^A) \\
= 0 & \text{for } x_1^A = x_1(w, p)
\end{cases}
\]

i.e., \( \phi(w, p, x_1^A) \) attains a minimum (= 0) over \((w, p)\) at all \( x_1^A = x_1(w, p) \). By the second order condition for an interior minimum,
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\[ \left[ \frac{\partial^2 \phi(w, p, x_i^A)}{\partial w \partial p} \right]_{(N+1) \times (N+1)} \equiv \left[ \frac{\partial^2 \pi(w, p)}{\partial w \partial p} \right] - \left[ \frac{\partial^2 \pi(w, p, x_i^A)}{\partial w \partial p} \right] \quad (2.17) \]

is positive semidefinite at \( x_i^A = x_i(w, p) \). (2.17) and Hotelling’s Lemma establish (2.14).

Samuelson’s Le Chatelier Principle (2.13)/(2.14) has often been given the following dynamic interpretation: assuming that the difference between short-run, intermediate run and long-run equilibrium can be characterized in terms of the number of inputs that can be adjusted within these time frames, the magnitude of the firm’s response \( \Delta x_i \) (or \( \Delta y \)) to a given change in price \( w_i \) (or \( p \)) increases over time, and the sign of these responses does not vary with the time frame.

However this characterization of dynamics in terms of a series of static models with a varying number of fixed inputs is unsatisfactory. In general dynamic behavior must be analyzed in terms of truly dynamic models.

For example, it often appears that an increase in price for beef output leads to a short-run decrease in beef output and a long-run increase in output, which contradicts the dynamic interpretation of Samuelson’s Le Chatelier Principle (2.13). This can be explained in terms of the dual role of cattle as output and as capital input to future production of output: a long-run increase in output generally requires an increase in capital stock, and this can be achieved by a short-run decrease in output (Jarvis 1974). This illustrates the following point: a series of static models with a varying number of fixed inputs completely ignores the intertemporal decisions associated with the accumulation of capital (durable goods).

Nevertheless, versions of profit functions conditional on the levels of certain inputs can be useful in applied work. The following restricted dual profit function is conditional on the level of capital stocks \( K \):

\[ \pi(w, p, K) \equiv \max_{x \geq 0} \left\{ pf(x, K) - \sum_{i=1}^{N} w_i x_i \right\} . \quad (2.18) \]

\( \pi(w, p, K) \) has the same properties in its price space \( (w, p) \) as does the unrestricted profit function \( \pi(w, p) \), which implicitly treats capital (or services from capital) as a freely adjustable input. In addition \( \frac{\partial \pi(w, p, K)}{\partial K} \) measures the shadow price of capital, and twice differentiability of \( \pi(w, p, K) \) and Hotelling’s Lemma establish the following reciprocity conditions:

\[ \frac{\partial y(w, p, K)}{\partial K} = \frac{\partial}{\partial p} \left( \frac{\partial \pi(w, p, K)}{\partial K} \right) \]
\[ \frac{\partial x_i(w, p, K)}{\partial K} = -\frac{\partial}{\partial w_i} \left( \frac{\partial \pi(w, p, K)}{\partial K} \right) \quad i = 1, \ldots, N. \quad (2.19) \]

There are three major advantages to specifying a restricted dual profit function. First, a restricted profit function \( \pi(w, p, K) \) is consistent with short-run equilibrium for a variety of dynamic models that dichotomize inputs as being either perfectly
variable or quasi-fixed in the short-run. Second, it is more realistic to assume that firms are in a short-run equilibrium rather than in a long-run equilibrium, and mis-specifying an econometric model as long-run equilibrium (e.g. using an unrestricted profit function $\pi(w, p)$) implies that the resulting estimators of long-run equilibrium effects of policy are unreliable. Third, long-run equilibrium effects can sometimes be inferred correctly from the estimates of $\pi(w, p, K)$ using the following first order condition for a long-run equilibrium level of capital stock $K^{**}$:

$$\frac{\partial \pi(w, p, K^{**})}{\partial K} = w_K \quad \text{or} \quad \frac{\partial \pi(w, p, K^{**})}{\partial K} = (r + \delta)p_K \quad (2.20)$$

where $w_K \equiv$ rental price of capital, $p_K \equiv$ asset price of capital, $r \equiv$ an appropriate discount rate and $\delta \equiv$ rate of depreciation for capital. After estimating $\pi(w, p, K)$ we would solve (2.20) for $K^{**}$.

### 2.6 Application of dual profit functions in econometrics: I

As in the use of a cost function $c(w, y)$, the above theory is usually applied by first specifying a functional form $\psi(w, p)$ for a profit function $\pi(w, p)$ and differentiating $\psi(w, p)$ with respect to $(w, p)$ to obtain the estimating equations

$$
y = \frac{\partial \psi(w, p)}{\partial p}, \quad x_i = -\frac{\partial \psi(w, p)}{\partial w_i} \quad i = 1, \ldots, N \quad (2.21)
$$

(employing Hotelling’s Lemma). Then the symmetry restrictions $\frac{\partial y}{\partial w_i} = -\frac{\partial x_j}{\partial p}$, $\frac{\partial x_i}{\partial w_j} = \frac{\partial x_j}{\partial w_i}$, $(i, j = 1, \ldots, N)$ are tested and $\left[ \frac{\partial^2 \psi}{\partial w \partial p} \right]_{(N+1) \times (N+1)}$ is checked for positive semidefiniteness. For example, a profit function could be postulated as

$$\pi = \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} \sqrt{w_i} \sqrt{w_j} + \sum_{j=1}^{N} a_{0j} \sqrt{w_j} \sqrt{p} + a \sqrt{p}$$

---

2 In general $\frac{\partial \pi(w, p, K)}{\partial K}$ cannot be measured directly from the estimating equations (a) $y = \frac{\partial \pi(w, p, K)}{\partial p}$, (b) $x = -\frac{\partial \pi(w, p, K)}{\partial w_i}$, e.g. when $\pi = K \sum_i \sum_j a_{ij} \sqrt{v_i} \sqrt{v_j} + a_K K$, $(v = (w, p))$. Estimates of $\frac{\partial \pi(w, p, K)}{\partial K}$ can be obtained either by estimating $\pi = \pi(w, p, K)$ jointly with (a)–(b) and then calculating directly $\frac{\partial \pi(w, p, K)}{\partial K}$, or by estimating (a)–(b) and then calculating $\frac{\partial \pi(w, p, K)}{\partial K} = p \frac{\partial \psi(w, p)}{\partial K} - \sum_{i=1}^{N} w_i \frac{\partial x_i}{\partial w_i}$, this last equation is derived as follows, $\pi(\lambda w, \lambda p, K) = \lambda \pi(w, p, K) \Rightarrow \frac{\partial \pi(\lambda w, \lambda p, K)}{\partial K} = \lambda \frac{\partial \pi(w, p, K)}{\partial K} \Rightarrow \frac{\partial \pi(w, p, K)}{\partial K} = p \frac{\partial \psi(w, p)}{\partial p} \frac{\partial \pi(w, p, K)}{\partial K} + \sum_{i=1}^{N} w_i \frac{\partial x_i}{\partial w_i} \frac{\partial \pi(w, p, K)}{\partial K}$ (Euler’s theorem), and then applying (2.19).
(this is a Generalized Leontief functional form, which imposes homogeneity of degree one in prices on the profit function). This leads to the following equations for estimation:

\[ y = a + \sum_{j=1}^{N} a_{0j} \left( \frac{w_j}{p} \right)^{1/2} \]

\[ x_i = \sum_{j=1}^{N} a_{ij} \left( \frac{w_j}{w_i} \right)^{1/2} + a_{i0} \left( \frac{p}{w_i} \right)^{1/2} \quad i = 1, \ldots, N \]

(apply Hotelling’s Lemma). Here the symmetry restrictions are expressed as \( a_{ij} = a_{ji} \) \((i, j = 1, \ldots, N)\) and \( a_{0i} = -a_{i0} \) \((i = 1, \ldots, N)\). If and only if the symmetry and second order conditions are satisfied, then equations (2.21) can be interpreted as being derived from a profit function \( \pi(w, p) \) for a producer showing static, competitive profit maximizing behavior. Similar comments apply to the estimation of a restricted profit function \( \pi(w, p, K) \).

Here we can note three major advantages of this approach to the modeling of producer behavior. First, it enables us to specify systems of output supply and factor demand equations that are consistent with profit maximization and with a general specification of technology. This cannot be achieved by estimating a production function directly together with first order conditions for profit maximization. Second, modeling a dual profit function explicitly allows for the endogeneity of output levels to the producer. This is in contrast to cost functions \( c(w, y) \) where output generally is treated as exogenous. Third, aggregation problems are likely to be less severe for unrestricted dual profit functions than for dual cost functions. If all firms \( f = 1, \ldots, F \) (a constant number of firms over time) face identical prices \((w, p)\) then it is clear that a profit function can be well defined for aggregate market data:

\[ \sum_{f=1}^{F} \pi_f(w, p) = \Phi(w, p) = \Pi(w, p). \]

In the special case of profit maximization at identical prices, a cost function also is well defined for data aggregated over firms even though the output level \( y_f \) varies over firms (we shall use this in a later lecture); but in this case we may as well estimate \( \Pi(w, p) \) directly.

The disadvantage of estimating a profit function \( \pi(w, p) \) rather than a cost function \( c(w, y) \) is that the former imposes stronger behavioral assumptions which are often very unrealistic. For example, at the time of production decisions, farmers generally have better knowledge of input prices than of output prices forthcoming at the time of marketing in the future. Thus risk aversion and errors in forecasting prices are more likely to influence the choice of output levels rather than to contradict the hypothesis of cost minimization. In addition in the case of food retail industries, hypothesis of oligopoly behavior plus competitive cost minimization may be more realistic than the hypothesis of competitive profit maximization.
2.7 Industry profit functions and entry and exit of firms

As mentioned above, an industry profit function is well defined provided that all firms face identical prices \((w, p)\) and the composition of the industry in terms of firms does not change. In this case the industry profit function is simply the sum of the profit functions of the firms \(f = 1, \ldots, F\): \(\Pi(w, p) = \sum_{f=1}^{F} \pi_f(w, p)\) for all \((w, p)\).

A more realistic assumption is that changes in prices \((w, p)\) induce some established firms to exit the industry and some new firms to enter the industry. Suppose that there is free entry and exit to the industry (all firms in the industry earn non-negative profits because all firms that would earn negative profits at \((w, p)\) are able to exit the industry, and all firms excluded from the industry are also at long-run equilibrium). Then the industry profit function inherits essentially the same properties as the individual firm’s profit function \(\pi_f(w, p)\).

Proposition 2.1 provides a characterization of the industry profit function \(\pi(w, p)\), assuming (a) competitive behavior, (b) both input prices \(w = (w_1, \ldots, w_N)\) and output prices \(p\) are exogenous, and (c) a continuum of firms and free entry/exit to the industry.

The role of the assumption of a continuum of firms deserves comment. As noted by Novshek and Sonnenschein (1979) in the case of marginal consumers, an infinitesimal change in price will lead to entry/exit behavior only in the case of a continuum of agents. Therefore it is necessary to assume that there exists a continuum of firms. Industry profits are calculated by integrating over the continuum of firms in the industry:

\[
\Pi(w, p) = \int_{1}^{f^m(w, p)} \pi(w, p, f) \rho(f) \, df
\]

where \(\rho(f)\) is the density of firms \(f\). \(\pi(w, p, f)\) denotes the individual firm’s profit function conditional on the firm being in the industry, and industry profits are obtained by integrating over those firms in the industry given prices \((w, p)\) and free entry/exit.

Adapting the arguments of Novshek and Sonnenschein, the assumption of a continuum of firms also establishes the differentiability of the industry profit function \(\Pi(w, p)\), industry factor demands \(X(w, p)\) and industry output supplies \(Y(w, p)\) with free entry/exit. The argument can be outlined as follows. Since the individual firm’s profit function \(\pi(w, p, f)\) is conditional on firm \(f\) remaining in the industry, it is reasonable to assume that \(\pi(w, p, f)\) is twice differentiable in \((w, p)\), i.e., \(\pi(w, p, f)\) is not kinked at \(\pi = 0\) due to exit from the industry. \(\pi(w, p, f)\) is differentiable in \(f\) and the derivative \(\pi_f(w, p, f) < 0\) assuming a continuum of firms indexed in descending order of profits. Then (by the implicit function theorem) a marginal firm \(f^m = f^m(w, p)\) is defined implicitly by the zero profit condition \(\pi(w, p, f^m) = 0\), and \(f^m(w, p)\) is differentiable. Under these assumptions it can easily be shown that the industry profit function with free entry/exit is differentiable, and its derivatives \(\Pi_w(w, p), \Pi_p(w, p)\) can be calculated by applying Leibnitz’s
rule to the above equation for industry profits (note that the zero profit condition \( \pi(w, p, f^m) = 0 \) established Hotelling’s Lemma at the industry level). A similar procedure establishes the differentiability of industry factor demands \( X(w, p) \) and output supplies \( Y(w, p) \) with free entry/exit.

**Proposition 2.1.** Assume that the industry consists of a continuum of firms such that each individual firm’s profit function \( \pi(w, p, f) \) is twice differentiable in \((w, p)\), differentiable in \( f \), \( \pi_f(w, p, f) < 0 \), and is linear homogeneous and convex in \((w, p)\) and satisfies Hotelling’s Lemma. Also assume \( \pi(w, p, f^m) = 0 \) for a marginal firm \( f^m \). Then \( \pi(w, p) \) is linear homogeneous and convex in \((w, p)\).

Moreover, it also satisfies Hotelling’s Lemma, i.e.,

\[
-X_i(w, p) = \frac{\partial \Pi(w, p)}{\partial w_i} \quad i = 1, \ldots, N
\]

\[
Y_k(w, p) = \frac{\partial \Pi(w, p)}{\partial p_k} \quad k = 1, \ldots, N
\]

(2.24)

and industry derived demands \( X(w, p) \) and output supplies \( Y(w, p) \) are differentiable.

**Proof.** Given prices \((w, p)\) and free entry/exit, index firms in the industry in descending order of profits. Industry profits can be calculated as

\[
\Pi(w, p) = \int_{f^m(w, p)}^f \pi(w, p, f) \rho(f) \, df
\]

(2.25)

where \( \rho(f) \) is the density of firms \( f \) and a marginal firm \( f^m \) satisfies the zero profit condition

\[
\pi(w, p, f^m) = 0.
\]

(2.26)

Assuming \( \pi(w, p, f) \) differentiable in \((w, p, f)\) and \( \pi_f(w, p, f) < 0 \), the zero profit condition (2.26) establishes (using the implicit function theorem) \( f^m = f^m(w, p) \) is defined and differentiable. Differentiability of \( \pi(w, p, f) \) and \( f^m(w, p) \) establishes (using (2.25) \( \Pi(w, p) \) is differentiable). Applying Leibnitz’s rule to (2.25), Pro:2.1

\[
\frac{\partial \Pi(w, p)}{\partial w_i} = \int_{f^m(w, p)}^f \frac{\partial \pi(w, p, f)}{\partial w_i} \rho(f) \, df + \pi(w, p, f^m(w, p)) \rho(f^m(w, p)) \frac{\partial f^m(w, p)}{\partial w_i}
\]

(2.27)

\[
\frac{\partial \Pi(w, p)}{\partial p_k} = \int_{f^m(w, p)}^f \frac{\partial \pi(w, p, f)}{\partial p_k} \rho(f) \, df + \pi(w, p, f^m(w, p)) \rho(f^m(w, p)) \frac{\partial f^m(w, p)}{\partial p_k}
\]

(2.28)
2.8 Application of dual profit functions in econometrics: II

Our development of Hotelling’s Lemma when the number of firms is variable to the industry has important implications for empirical studies. An industry profit function \( \Pi(w, p) \) satisfies Hotelling’s Lemma in two extreme cases: the number of firms in the industry is fixed (the standard case) or there is free entry/exit to the industry (Proposition 2.1). In intermediate cases, where the number of firms is variable but entry/exit is not instantaneous and costless, the Lemma does not apply. This can be seen from equations (2.29) and (2.30) in the proof of Proposition 2.1: applying Leibnitz’s rule to the integral for industry profits over a continuum of firms with entry/exit

\[
\Pi(w, p) = \int_1^{f^m(w, p)} \pi(w, p, f) \rho(f) \, df
\]

yields Hotelling’s Lemma only if there exist marginal firms earning zero profits, i.e., \( \pi(w, p, f^m(w, p)) = 0 \) (assume for the sake of argument that \( f^m(w, p) \) is differentiable in the intermediate case). This zero profit condition is characteristic of free entry/exit with a continuum of firms but it is not characteristic of the intermediate case.

The usual assumptions that are acknowledged in standard applications of Hotelling’s Lemma to industry-level data are that each firm in the industry shows static competitive profit maximizing behavior (conditional on the firm being in the industry). In addition we must generally add the restrictive assumption that the composition
of the industry does not change over the time period of the data or there is free entry/exit to the industry.

Nevertheless there is at least in principle a simple procedure for avoiding this additional restrictive assumption: the industry profit function can be defined explicitly as conditional upon the number of firms of each different type. For example, suppose that an industry consists of two homogeneous types of firms in variable quantities $F_1$ and $F_2$. The industry profit function can be written as $\Pi(w, p, F_1, F_2)$, where $F_1$ and $F_2$ are specified as parameters along with $(w, p)$. Then $X(w, p, F_1, F_2) = -\Pi_w(w, p, F_1, F_2)$, $Y(w, p, F_1, F_2) = \Pi_p(w, p, F_1, F_2)$ by standard argument (e.g. Bliss). Of course in practice reasonable data on the number of firms by type is not always available, but whenever possible such modifications seem likely to improve the specification of the model.

Thus, if we have time series data on the number of firms $F_1$ and $F_2$ in the two classes as well as data on total output, total inputs and prices, then we can postulate an industry profit function $\Pi(w, p, F_1, F_2)$ conditional on $F_1$, $F_2$ and apply Hotelling’s Lemma to obtain the estimating equations

$$Y = \frac{\partial \Pi(w, p, F_1, F_2)}{\partial p}$$

$$X_i = -\frac{\partial \Pi(w, p, F_1, F_2)}{\partial w_i} \quad i = 1, \ldots, N.$$  \hspace{1cm} (2.31)

If there is free entry/exit to the industry, then (by 2.1) parameters $F_1$ and $F_2$ drop out of the system of estimating equations (2.31)—in this manner the assumption of long-run industry equilibrium is easily tested. If there is not free entry/exit and if the number of firms $F_1$ and $F_2$ vary over time, then the parameters $F_1$ and $F_2$ are significant in equations (2.31). The stocks of firms $F_1$ and $F_2$ at any time $t$ probably can be approximated as predetermined at time $t$ (i.e., the number of firms is essentially inherited from the past, given substantial delays in entry and exit). Then $F_{1,t}$ and $F_{2,t}$ do not necessarily covary with the disturbance terms at time $t$ for equations (2.31), so that equations (2.31) may be estimated consistently even if there is costly entry/exit to the industry.

However for policy purposes it may be desirable to estimate (2.31) jointly with equations of motion for the number of firms:

$$F_{1,t+1} - F_{1,t} = F_1(w_t, p_t, F_{1,t}, \cdots)$$

$$F_{2,t+1} - F_{2,t} = F_2(w_t, p_t, F_{2,t}, \cdots)$$  \hspace{1cm} (2.32)

Equations (2.31) can indicate the short-run impact of price policies or industry output and inputs levels $(Y, X)$, i.e., the impact of the price policies before there is an adjustment in the number of firms.

Equations (2.31) and (2.32) jointly indicate intermediate and long-run impacts of price policies. For example, at a static long-run equilibrium there is no exit/entry to the industry, so the long-run equilibrium numbers of firms $F_1^{**}$, $F_2^{**}$ for any $(w, p)$ can be calculated from (2.32) by solving the implicit equations $F_1(w_t, p_t, F_{1,t}^{**}) = \cdots$.
0, \( F_2(w_t, p_t, F_{2*t}) = 0 \). Obvious difficulties here are (a) problems in specifying the dynamics of entry and exit (equations (2.32)) correctly, and (b) dangers in using (2.31) to extrapolate to long-run equilibrium numbers of firms \( F_{1**}, F_{2**} \) that are outside of the data set.

References

Chapter 3
Static Utility Maximization and Expenditure Constraints

Here we model both consumer and producer behavior subject to expenditure constraints. We begin with the case of consumer.

Consider a consumer maximizing utility \( u \) by allocating his income \( y \) among \( N \) commodities \( x = (x_1, \cdots, x_N) \), and denote his utility function as \( u = u(x) \). Assume that the consumer takes commodity prices \( p = (p_1, \cdots, p_N) \) as given and solves the following static competitive utility maximization problem:

\[
\max_{x \geq 0} u(x) = u(x^*) \\
\text{s.t. } \sum_{i=1}^N p_i x_i \leq y \tag{3.1}
\]

The maximum utility \( V \equiv u(x^*) \) to problem (3.1) depends on prices and income \((p, y)\) and the consumer’s utility function \( u = u(x) \). The corresponding relation \( V = V(p, y) \) between maximum utility and prices and incomes is denoted as the consumer’s dual indirect utility function.

A necessary condition for utility maximization (3.1) is that the consumer attains the utility level \( u(x^*) \) at a minimum cost. In other words, if \( x^* \) does not minimize the cost \( \sum_{i=1}^N p_i x_i \) subject to \( u(x) = u(x^*) \), then a higher utility level \( u > u(x^*) \) can be attained at the same cost \( \sum_{i=1}^N p_i x_i^* = y \) (for this result we only need to assume local nonsatiation of \( u(x) \) in neighborhood of \( x^* \)). Thus a solution \( x^* \) to the utility maximization problem (3.1) also solves the following cost minimization problem when the exogenous utility level \( u \) is equal to \( u(x^*) \):

\[
\min_{x \geq 0} \sum_{i=1}^N p_i x_i = \sum_{i=1}^N p_i x_i^* \\
\text{s.t. } u(x) \geq u \tag{3.2}
\]

The minimum cost \( E = \sum_{i=1}^N p_i x_i^* \) to problem (3.2) depends on prices and utility level \((p, u)\) and the consumers utility function \( u(x) \). The corresponding relation
between minimum expenditure and prices and utility level, \( E = E(p, u) \), is denoted as the consumers dual expenditure function. Note that (3.2) is formally equivalent to the producer’s cost minimization problem \( \min_{x \geq 0} \sum_{i=1}^{N} p_{i} x_{i} \) s.t. \( f(x) \geq y \) (1.1). Thus the expenditure function \( E(p, u) \) inherits the essential properties (1.1) of the producers cost function \( c(w, y) \):

**Property 3.1.**

a) \( E(p, u) \) is increasing \((p, u)\).

b) \( E(p, u) \) is linear homogeneous in \( p \).

c) \( E(p, u) \) is concave in \( p \).

d) If \( E(p, u) \) is differentiable in \( p \), then

\[
x_{i}(p, u) = \frac{\partial E(p, u)}{\partial p_{i}} \quad i = 1, \cdots, N.
\]

(Shephard’s Lemma)

Also note that a sufficient condition for utility maximization (3.1) is that the consumer attains the utility level \( u(x^*) \) at a minimum cost equals to \( \sum_{i=1}^{N} p_{i} x_{i}^* \). In other words, if \( x^A \) solves (3.2) subject to \( u(x) = u^A \), then \( x^A \) also solves (3.1) subject to \( \sum_{i=1}^{N} p_{i} x_{i} = y \equiv \sum_{i=1}^{N} p_{i} x_{i}^A \).

**Proof.** Assume the that \( u(x) \) is continuous and \( x^A \) solves (3.2) at the exogenously determined utility level \( u \equiv u^A \). Now suppose that \( x^* \) rather than \( x^A \) solves (3.1) at the exogenously determined expenditure level \( y^A = \sum_{i=1}^{N} p_{i} x_{i}^A > 0 \). This would imply \( u(x^*) > u(x^A) \) and (given local nonsatiation) \( \sum_{i=1}^{N} p_{i} x_{i}^* = \sum_{i=1}^{N} p_{i} x_{i}^A \). Then continuity of \( u(x) \) would imply that there exists an \( \tilde{x} \) in the neighborhood of \( x^* \) such that \( u(x^*) \geq u(\tilde{x}) \geq u(x^A) \) and \( \sum_{i=1}^{N} p_{i} \tilde{x}_{i} < \sum_{i=1}^{N} p_{i} x_{i}^A \), which contradicts the assumption \( x^A \) solves (3.2). Therefore \( x^A \) solves (3.2) implies \( x^A \) solves (3.1).

Thus \( x^A \) solves (3.1) if and only if \( x^A \) solves (3.2) subject to \( u \equiv u(x^A) \). This implies that the restrictions placed on Marshallian consumer demands \( x = x(p, y) \) (corresponding to problem (3.1)) by the hypothesis of utility maximization (3.1) can be analyzed equivalently in terms of the restrictions placed on Hicksian consumer demands \( x = x^h(p, u) \) (corresponding to problem (3.2)) by the hypothesis of cost minimization (3.2). In other words, the hypothesis of cost minimization (3.2) exhausts the restrictions placed on Marshallian demands \( x = x(p, y) \) by the hypothesis of utility maximization. Properties 3.1 of \( E(p, u) \) imply

**Property 3.2.**

a) \( x(p, u) \) are homogenous of degree 0 in \( p \). i.e. \( x(\lambda p, u) = x(p, u) \) for all scalar \( \lambda > 0 \).
And properties 3.2 exhausts the implications of cost minimization for the (local) properties of Hicksian demands $x^h(p,u)$ (the proof is the same as in the case of cost minimization by a producer). Therefore, by the proof immediately above, 3.2 also exhausts the implications of utility maximization (3.1) for the (local) properties of Marshallian demands $x(p,y)$, where 3.2 is evaluated at a utility maximization $u = u(x(p,y))$. Of course the characterization of utility maximization in terms of (3.2) is not immediately useful empirically in the sense that utility level $u$ is not observed (nor is $u$ exogenous to the consumer).

This relation between utility maximization and cost minimization is very different from the relation between profit maximization and cost minimization for the producer: profit maximization implies but is not equivalent to cost minimization in any sense. The explanation is that in the consumer case, in contrast to the producer case, maximization is subject to an expenditure constraint defined over all commodities.

### 3.1 Properties of $V(p, y)$

**Property 3.3.**

a) $V(p, y)$ is decreasing in $p$ and increasing in $y$.

b) $V(p, y)$ is homogenous of degree 0 in $(p, y)$, i.e. $V(\lambda p, \lambda y) = V(p, y)$ for all scalar $\lambda > 0$.

c) $V(p, y)$ is quasi-convex in $p$. i.e. $\{p : V(p, y) \leq k\}$ is a convex set for all scalar $k \geq 0$. See Figure 3.1

d) If $V(p, y)$ is differentiable in $(p, y)$, then

$$x_i(p, y) = \frac{\partial V(p, y)/\partial p_i}{\partial V(p, y)/\partial y} \quad i = 1, \ldots, N$$

(Roy’s Theorem)

*Proof.* Properties 3.3.a–b follows simply from the definition of the consumer’s maximization problem (3.1). For a proof of 3.2.c see Varian (1992, pp. 121–122). In order to prove 3.3.d note that, if $u^*$ is the maximum utility for (3.1) given parameters $(p, y)$, then $u^* = V(p, y^*)$ where $y^* \equiv E(p, u^*)$, i.e. $y^*$ is the minimum expenditure necessary to attain a utility level $u^*$ given prices $p$. Total differentiating this
Roy’s Theorem and $V(p, y)$ quasi-convex in $p$ can also be derived as follows. Since $V(p, y) \equiv \max_x u(x)$ s.t. $px = y$,

$$V(p, y) - u(x) \geq 0 \quad \text{for all } (p, y, x) \text{ such that } px = y$$  \hspace{1cm} (3.4)

or equivalently

$$V\left(p, \sum_{i=1}^{N} p_i x_i\right) - u(x) \geq 0 \quad \text{for all } (p, x)$$  \hspace{1cm} (3.5)

where $y$ is now defined implicitly as $y = \sum_{i=1}^{N} p_i x_i$. By (3.5),

$$G(p, x) \equiv V\left(p, \sum_{i=1}^{N} p_i x_i\right) - u(x) \geq 0 \quad \text{for all } p$$

$$G(p, x) \equiv V\left(p, \sum_{i=1}^{N} p_i x_i\right) - u(x) = 0 \quad \text{at } p \text{ such that } x = x(p, y)$$  \hspace{1cm} (3.6)

for $y = \sum_{i=1}^{N} p_i x_i$

In other words, given that $x^A$ solves (3.1) conditional on $(p^A, y^A)$, then $G(p, x^A)$ attains a global minimum over $p$ at $p^A$. This implies the following first and second order conditions for maximization:

$$\frac{\partial G(p^A, x^A)}{\partial p_i} = 0 \quad i = 1, \ldots, N$$

$$\left[\frac{\partial^2 G(p^A, x^A)}{\partial p_i \partial p_j}\right]_{N \times N} \text{ is symmetric positive semidefinite.}$$  \hspace{1cm} (3.7)

Since $G(p, x) \equiv V\left(p, \sum_{i=1}^{N} p_i x_i\right) - u(x)$, conditions (3.7) imply
3.2 Corresponding properties of \( x(p, y) \) solving problem (3.1)

\[
\frac{\partial G(p, x)}{\partial p_i} = \frac{\partial V(p, y)}{\partial p_i} + \frac{\partial V(p, y)}{\partial y} x_i = 0 \quad i, j = 1, \cdots, N \quad (3.8a)
\]

\[
\frac{\partial^2 G(p, x)}{\partial p_i \partial p_j} = \frac{\partial^2 V(p, y)}{\partial p_i \partial p_j} + \frac{\partial^2 V(p, y)}{\partial p_i \partial y} x_j + \frac{\partial^2 V(p, y)}{\partial y \partial p_i} x_i + \frac{\partial^2 V(p, y)}{\partial y \partial y} x_j x_i \quad (3.8b)
\]

(3.8a) is Roy’s Theorem. \( \left[ \frac{\partial^2 G(p, x)}{\partial p \partial p} \right]_{N \times N} \) is symmetric positive semidefinite (3.7) implies semidefinite—these are the second order restrictions implied by \( V(p, y) \) quasi-convex in \( p \).

### Property 3.4.

a) \( x(p, y) \) is homogeneous of degree 0 in \( (p, y) \), i.e. \( x(\lambda p, \lambda y) = x(p, y) \) for all scalar \( \lambda \geq 0 \).

b) \[
\frac{\partial x_i(p, y)}{\partial p_j} = \frac{\partial x^h_i(p, u^*)}{\partial p_j} - \frac{\partial x_i(p, y)}{\partial y} x_j(p, y) \quad i, j = 1, \cdots, N \quad (Slutsky equation)
\]

(3.9) is symmetric negative semidefinite.

**Proof.** Property 3.4.a is obviously true. In order to prove 3.4.b, let \( u^* \) be the maximal utility for problem (3.1) conditional on \( (p, y) \). Then we have proved the following identity (see pages 27–28): \( x_i(p, y) \equiv x^h_i(p, u^*) \) where \( y \equiv E(p, u^*) \), i.e.

\[
x^h_i(p, u^*) = x_i(p, E(p, u^*)) \quad i = 1, \cdots, N \quad (3.9)
\]

In words, the Hicksian and Marshallian demands \( x^h(p, u) \) and \( x(p, y) \) are equal when \( x^h(p, u) \) are evaluated at a utility level \( u = u^* \) solving (3.1) for prices \( p \) and a given income \( y \) and \( x(p, y) \) are evaluated at \( p \) and a level \( y \equiv E(p, u^*) \) solving (3.2) given \( p \) and the level \( u = u^* \). Differentiating (3.9) with respect to \( p_j \) (holding utility level \( u^* \) constant) yields

\[
\frac{\partial x^h_i(p, u^*)}{\partial p_j} = \frac{\partial x_i(p, y)}{\partial p_j} + \frac{\partial x_i(p, y)}{\partial y} \frac{\partial E(p, u^*)}{\partial p_j} \quad i, j = 1, \cdots, N. \quad (3.10)
\]

Since (using Shephard’s Lemma) \( \frac{\partial E(p, u^*)}{\partial p_j} = x_j(p, u^*) = x_j(p, y) \) \( (j = 1, \cdots, N) \) and \( \left[ \frac{\partial x(p, u^*)}{\partial p} \right]_{N \times N} \) is symmetric negative semidefinite from 3.2.b, (3.10) establishes 3.4.b. □
The integrability problem has traditionally been prosed as follows: given a set of Marshallian consumer demand relations \( x = x(p, y) \), what restrictions on \( x(p, y) \) exhaust the hypothesis of competitive utility maximizing behavior by the consumer? We can now construct an answer as follows. Since utility maximization (3.1) and expenditure minimization subject to \( u(x) = u^* \) are equivalent (see pages 27–28), we can rephrase the question as: what restrictions on \( x(p, y) \) exhaust the hypothesis of competitive cost minimizing behavior by the consumer? Using the Slutsky equation 3.4.b we can recover the matrix of substitution effects \( \partial x^h(p, u^*) / \partial p \) for the cost minimization demands \( x^h(p, u^*) \) from the Marshallian demands \( x(p, y) \). \( x^h(p, u^*) \) homogeneous of degree 0 in \( p \) and symmetry of \( \partial x^h(p, u^*) / \partial p \) imply the set of differential equations \( x^h(p, u^*) = \partial E(p, u^*) / \partial p_i, i = 1, \cdots, N \) (Shephard’s Lemma) (see page 4). By the Frobenius theorem, these differential equations can be integrated up to a macrofunction \( E(p, u^*) \) if and only if \( \partial x^h(p, u^*) / \partial p \) is symmetric. Furthermore \( \partial x^h(p, u^*) / \partial p \) negative semidefinite implies (by Shephard’s Lemma) \( \partial^2 E(p, u^*) / \partial p \partial p \) negative semidefinite, which in turn implies \( E(p, u^*) \) concave and \( u(x) \) quasi-concave at \( x(p, y) \). Thus Marshallian demand relations \( x = x(p, y) \) can be interpreted as being derived from competitive utility maximizing behavior if and only if

\[
\begin{bmatrix}
\partial x(p, y) / \partial p \\
\partial x(p, y) / \partial y
\end{bmatrix}_{N \times 1} \equiv \begin{bmatrix}
\partial x^h(p, u^*) / \partial p \\
\partial x^h(p, u^*) / \partial y
\end{bmatrix}_{N \times N}
\]

is symmetric negative semidefinite,

\[
x(p, y) \text{ homogenous of degree } 0 \text{ in } (p, y).
\]

3.3 Application of dual indirect utility functions in econometrics

The above theory is usually applied by first specifying a functional form \( \psi(p, y) \) for the indirect utility function \( V(p, y) \) and differentiating \( \psi(p, y) \) with respect to \( (p, y) \) in order to obtain the estimating equations

\[
x_i = -\partial \psi(p, y) / \partial p_i \frac{\partial \psi(p, y)}{\partial y} \quad i = 1, \cdots, N
\]

\[x(p, y) \text{ homogenous of degree } 0 \text{ in } (p, y) \] implies \( x^h(p, u^*) \) homogenous of degree 0 in \( p \), since \( x(p, y) \equiv x^h(p, V(p, y)) \).
(using Roy’s Theorem). \( \psi(p, y) \) is usually specified as being homogeneous of degree 0 in \((p, y)\).

The symmetry conditions

\[
\frac{\partial^2 \psi(p, y)}{\partial p_i \partial p_j} = \frac{\partial^2 \psi(p, y)}{\partial p_j \partial p_i}, \quad \frac{\partial^2 \psi(p, y)}{\partial p_i \partial y} = \frac{\partial^2 \psi(p, y)}{\partial y \partial p_i}
\]

\(i, j = 1, \ldots, N\) (3.13)

are satisfied if and only if corresponding Hicksian demand relations \( x^h_i(p, u^*) = \frac{\partial E(p, u^*)}{\partial p_i} (i = 1, \ldots, N) \) integrate up to a macrofunction \( E(p, u^*) \).

**Proof.** By the Frobenius theorem, the Hicksian demands \( x^h_i(p, u^*) = \frac{\partial E(p, u^*)}{\partial p_i} (i = 1, \ldots, N) \) integrate up to a macrofunction if and only if \( x^h_i(p, u^*)/\partial p_j = x^h_j(p, u^*)/\partial p_i (i, j = 1, \ldots, N) \). Differentiating \( x_i = \frac{\partial V(p, y)}{\partial p_i} \) (Roy’s Theorem) with respect to \((p, y)\) and substituting into the Slutsky equation 3.4.b yields

\[
-\frac{\partial x^h_i(p, u^*)}{\partial p_j} = \frac{V_{p_i pj} V_y - V_{pj} V_{pj} V_{pi}}{V_y V_y} + \frac{V_{pj} V_y V_{pj} - V_{pj} V_{pj}}{V_y V_y} \cdot \left( \frac{V_{pj} V_y}{V_y} \right)
\]

(3.14)

where \( V_{p_i pj} = \frac{\partial}{\partial p_j} \left( \frac{\partial V(p, y)}{\partial p_i} \right) \), \( V_{pj} = \frac{\partial}{\partial p_j} \left( \frac{\partial V(p, y)}{\partial y} \right) \), \( V_{pj} V_{pi} = \frac{\partial}{\partial p_j} \left( \frac{\partial V(p, y)}{\partial p_i} \right) \) etc. Inspection of (3.14) shows that \( \partial x^h_i(p, u^*)/\partial p_j = \partial x^h_j(p, u^*)/\partial p_i \) if and only if \( V_{p_i pj} = V_{pj} V_{pi}, V_{pj} = V_{pj} (i, j = 1, \ldots, N) \). \( \Box \)

Thus the hypothesis of competitive utility maximization is verified by (a) testing for the symmetry restrictions (3.13) and (b) checking the second order conditions (3.8b) at all data points \((p, y)\) or (equivalently) using Slutsky relations to recover the Hicksian matrix \( \partial x^h/\partial p \) and checking for negative semidefiniteness.

For example an indirect utility function could be postulated as having the functional form \( V = y/\sum_i \sum_j a_{ij} p_i^{1/2} p_j^{1/2} \) (a Generalized Leontief reciprocal indirect utility function with homothecity). Applying Roy’s Theorem leads to the following functional form for the consumer demand equations:

\[
x_i = \frac{y \sum_{j=1}^N a_{ij} (p_j / p_i)^{1/2}}{\sum_{j=1}^N \sum_{k=1}^N a_{jk} p_j^{1/2} p_k^{1/2}} \quad i = 1, \ldots, N.
\]

(3.15)

Here the symmetry restrictions (3.13) are expressed as \( a_{ij} = a_{ji} (i, j = 1, \ldots, N) \) which are easily tested. Equations (3.15) can be interpreted as being derived from an indirect utility function \( V(p, y) = y/\sum_i \sum_j a_{ij} p_i^{1/2} p_j^{1/2} \) for a consumer showing static competitive utility maximizing behavior if and only if the symmetry and second order conditions are satisfied.

The major advantage of this approach to modeling consumer behavior is that it permits the specification of a system of Marshallian commodity demand equations \( x = x(p, y) \) that are consistent with utility maximization and with a very general specification of the consumer’s utility function.\(^2\) In contrast, even if an accept-

\(\footnote{2}{The high degree of flexibility of} V(p, y) \text{ in representing utility functions (consumer preferences) } u(x) \text{ follow from the fact that the first and second derivatives of } V(p, y) \text{ determine the first and}
able measure of utility is defined and the first order conditions \( \frac{\partial u(x^*)}{\partial x_i} = \frac{p_i}{p_j} \) \((i, j = 1, \ldots, N)\) are estimated, the corresponding Marshallian demand equations \( x = x(p, y) \) can be recovered explicitly only under very restrictive functional forms for \( u(x) \) (e.g. Cobb-Douglas).

A serious problem with this and all other consumer demand models derived from microtheory (behavior of the individual consumer) is that the data is usually aggregated over consumers. Difficulties raised by the use of such market data will be discussed in a later lecture.

### 3.4 Profit maximization subject to budget constraints

The above model (3.1) of consumer behavior subject to a budget constraint is formally equivalent to the following model of producer behavior subject to a budget constraint

\[
\max_{x \geq 0} \left\{ pf(x) - \sum_{i=1}^{N} w_i x_i \right\} \equiv \pi(w, p, b) \quad \text{(3.16)}
\]

subject to

\[
\sum_{i=1}^{N} w_i x_i = b
\]

since (3.16) and

\[
\max_{x \geq 0} pf(x) \equiv R(w, p, b) \quad \text{(3.17)}
\]

have identical solutions \( x^* \) assuming that the budget constraint is binding. Here purchases of all inputs are assumed to draw upon the same total budget \( b \), and there is a single output with production function \( y = f(x) \). In this case the solution \( x^* \) to (3.16) also solves the cost minimization problem

\[
\min_{x \geq 0} w_i x_i \equiv c(w, y^*) \quad \text{(3.18)}
\]

subject to \( f(x) = y^* \)

where \( y^* = f(x^*) \). Note that cost minimization (conditional on \( y^* \)) is sufficient as well as necessary for a solution to (3.16) (see pages 27–28), in contrast to the case of profit maximization without a budget constraint.

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second derivatives of the corresponding utility function \( u(x) \) at \( x(p, y) \). This can be seen by differentiating the first order conditions \( \frac{\partial u(x^*)}{\partial x_i} = \frac{\partial V(p, y)}{\partial y} p_i \), \( \sum_{i=1}^{N} p_i x_i^* = y \) \((i = 1, \ldots, N)\) for utility maximization and employing Roy’s Theorem, in a manner analogous to the derivation of (1.6).
The envelop relations and second order conditions for (3.16) and (3.17) are easily derived as follows. (3.16) implies
\[ \psi(w, p, b, x) \equiv \pi(w, p, b) - \{pf(x) - wx\} \geq 0 \]
for all \((w, p, b, x)\) such that \(wx = b\) \hfill (3.19)

or, substituting \(wx\) for \(b\) in \(\pi(\cdot)\),
\[ \tilde{\psi}(w, p, x(w, p, b)) = 0 \]

Thus \(\tilde{\psi}(w, p, x(w^0, p^0, b^0))\) attains a minimum over \((w, p)\) at \((w^0, p^0)\). This implies the following first order conditions for a minimum:
\[ \frac{\partial \tilde{\psi}(w^0, p^0, x^0)}{\partial p} = \frac{\partial \pi(w^0, p^0, b^0)}{\partial p} - f(x^0) = 0 \quad i = 1, \cdots, N \]
\[ \frac{\partial \tilde{\psi}(w^0, p^0, x^0)}{\partial w_j} = \frac{\partial \pi(w^0, p^0, b^0)}{\partial w_j} + \frac{\partial \pi(w^0, p^0, b^0)}{\partial b} x^0_i + x^0_i = 0 \]

i.e. (3.16) implies
\[ y(w, p, b) = \frac{\partial \pi(w, p, b)}{\partial p} \]
\[ x_i(w, p, b) = -\frac{\partial \pi(w, p, b)/\partial w_i}{1 + \partial \pi(w, p, b)/\partial b} \quad i = 1, \cdots, N \]

Note that (3.21) reduces to Hotelling’s Lemma if \(\partial \pi(w, p, b)/\partial b = 0\), i.e. if the budget constraint is not binding. The second order conditions for a minimum of \(\tilde{\psi}(w, p, x(w^0, p^0, b^0))\) over prices \((w, p)\) are \([\tilde{\psi}_{w, p}(w, p, x^0)\] positive semidefinite. Twice differentiating the identity \(\tilde{\psi}(w, p, x) \equiv \pi(w, p, wx) - \{pf(x) - wx\}\) with respect to prices \((w, p)\) yields
\[ \frac{\partial^2 \tilde{\psi}(w, p, x)}{\partial w_i \partial w_j} = \frac{\partial^2 \pi(w, p, b)}{\partial w_i \partial w_j} x_j + \frac{\partial^2 \pi(w, p, b)}{\partial w_i \partial b} x_i + \frac{\partial^2 \pi(w, p, b)}{\partial b \partial w_j} x_i x_j \]
\[ \frac{\partial^2 \tilde{\psi}(w, p, x)}{\partial w_i \partial p} = \frac{\partial^2 \pi(w, p, b)}{\partial w_i \partial p} x_i \]
\[ \frac{\partial^2 \tilde{\psi}(w, p, x)}{\partial p \partial w_i} = \frac{\partial^2 \pi(w, p, b)}{\partial p \partial w_i} x_i \]
\[ \frac{\partial^2 \tilde{\psi}(w, p, x)}{\partial p \partial w_i} = \frac{\partial^2 \pi(w, p, b)}{\partial p \partial w_i} x_i \quad i, j = 1, \cdots, N \]
Thus profit maximization subject to a budget constraint \( wx = b \) implies that the \((N + 1)\)-dimensional matrix defined in (3.22) is symmetric positive semidefinite.

Likewise (3.17) implies

\[
\phi(w, p, x) \equiv R(w, p, wx) - pf(x) \geq 0 \quad \text{for all} \quad (w, p, x) \\
\phi(w, p, x(w, p, b)) = 0
\]  

(3.23)

Proceeding as above we obtain the envelop relations

\[
y = \frac{\partial R(w, p, b)}{\partial p} \\
x_i = -\frac{\partial R(w, p, b)/\partial w_i}{\partial R(w, p, b)/\partial b}
\]  

(3.24)

and second order conditions analogous to (3.22) symmetric positive semidefinite.

Nevertheless, assuming an expenditure constraint over all inputs and a single output, the simplest approach is to estimate a cost function \( c(w, y) \) using Shephard’s Lemma to obtain estimating equations

\[
x_i = \frac{\partial c(w, y)}{\partial w_i} \quad i = 1, \cdots, N.
\]  

(3.25)

Under the above assumptions the hypothesis of cost minimization conditional on \( y^* \) exhausts the implications of the hypothesis of profit maximization subject to a budget constraint, although of course this approach (3.25) still mis-specifies the maximization problem (3.16) by treating output \( y \) as exogenous.

As a second and more interesting example of modeling budget constraints in production, suppose that expenditures on only a subset of inputs are subject to a budget constraint, (e.g. different inputs may be purchased at different times and cash constraints may be binding only at certain times, or alternatively credit may be available for the purchase of some but not all inputs). In this case the firm solves the profit maximization problem

\[
\max_{x_A, x_B \geq 0} \left\{ \sum_{i=1}^{N_A} w_i^A x_i^A - \sum_{i=1}^{N_B} w_i^B x_i^B \right\} = \pi(w, p, b) \\
\text{s.t.} \quad \sum_{i=N_A+1}^{N_A+N_B} w_i^B x_i^B = b
\]  

(3.26)

(3.26) implies

\[
\gamma(w, p, x) = \pi(w, p, w_B x_B) - \{ pf(x_A, x_B) - w_A x_A - w_B x_B \} \\
\geq 0 \quad \text{for all} \quad (w, p, x) \\
= 0 \quad \text{for} \quad (w, p, x(w, p, b))
\]  

(3.27)
Proceeding as before leads to the envelop relations

\[ y = \frac{\partial \pi(w, p, b)}{\partial p} \]

\[ x_i^A = -\frac{\partial \pi(w, p, b)}{\partial w_i^A} \quad i = 1, \ldots, N \tag{3.28} \]

\[ x_i^B = -\frac{\partial \pi(w, p, b)}{1 + \partial \pi(w, p, b)/\partial b} \quad i = N_A + 1, \ldots, N_A + N_B \]

and to the second order relations

\[ \frac{\partial^2 \gamma(w, p, x)}{\partial w_i^B \partial w_j^B} = \frac{\partial^2 \pi(w, p, b)}{\partial w_i^B \partial w_j^B} + \frac{\partial^2 \pi(w, p, b)}{\partial b \partial w_i^B} x_j^B + \frac{\partial^2 \pi(w, p, b)}{\partial b \partial w_j^B} x_i^B \quad i, j = 1, \ldots, N_B \]

\[ \frac{\partial^2 \gamma(w, p, x)}{\partial w_i^A \partial w_j^A} = \frac{\partial^2 \pi(w, p, b)}{\partial w_i^A \partial w_j^A} x_j^B \quad i = 1 \ldots, N_B \quad j = 1, \ldots, N_A \]

\[ \frac{\partial^2 \gamma(w, p, x)}{\partial w_i^A \partial p} = \frac{\partial^2 \pi(w, p, b)}{\partial w_i^A \partial p} x_j^B \quad i = 1 \ldots, N_B \quad j = 1, \ldots, N_A \]

\[ \frac{\partial^2 \gamma(w, p, x)}{\partial w_j^A \partial p} = \frac{\partial^2 \pi(w, p, b)}{\partial w_j^A \partial p} \quad i = 1, \ldots, N_A \]

\[ \frac{\partial^2 \gamma(w, p, x)}{\partial p \partial p} = \frac{\partial^2 \pi(w, p, b)}{\partial p \partial p} \tag{3.29} \]

The \((N_A + N_B + 1)\)-dimensional matrix defined by (3.29) should be symmetric positive semidefinite, assuming profit maximization subject to a binding budget constraint \(w_B x_B = b\).

The above theory of profit maximization subject to a budget constraint can be applied by first specifying a functional form \(\psi(w, p, b)\) for the profit function \(\pi(w, p, b)\) and differentiating \(\psi(w, p, b)\) with respect to \((w, p, b)\) in order to obtain the estimating equation

\[ y = \frac{\partial \psi(w, p, b)}{\partial p} \]

\[ x_i^A = -\frac{\partial \psi(w, p, b)}{\partial w_i^A} \quad i = 1, \ldots, N_A \tag{3.30} \]

\[ x_i^B = -\frac{\partial \psi(w, p, b)}{1 + \partial \psi(w, p, b)/\partial b} \quad i = N_A + 1, \ldots, N_A + N_B \]
where only inputs $x_B$ are subject to a budget constraint $w_B x_B = b$. The symmetry conditions
$$\frac{\partial^2 \psi}{\partial w_i \partial w_j} = \frac{\partial^2 \psi}{\partial w_j \partial w_i}, \quad \frac{\partial^2 \psi}{\partial w_i \partial p} = \frac{\partial^2 \psi}{\partial w_j \partial p}, \quad (i, j = 1, \cdots, N_A + N_B)$$
imply that the output supply and factor demand equations (3.30) integrate up to a macrofunction $\psi(w, p, b)$, and satisfaction of the second order conditions (3.29) (replacing $\pi$ with $\psi$ in the derivatives) implies that $\psi(w, p, b)$ can be interpreted as a profit function $\pi(w, p, b)$ corresponding to the behavioral model (3.26). Note that models such as (3.30) may be useful in testing whether expenditures on a particular input draw on a binding budget, i.e. in testing whether the shadow price for the budget constraint $\partial \psi(w, p, b)/\partial b$ is significant in the demand equation for a particular input.

For example, the functional form for $\pi(w, p, b)$ could be hypothesized as
$$\pi = b \left( \sum_i \sum_j a_{ij} w_i^{-\frac{1}{2}} w_j^{-\frac{1}{2}} + \sum_i a_{0i} p^{\frac{1}{2}} w_i^{\frac{1}{2}} \right)$$
(3.31)
(a Generalized Leontief functional form with constant returns to scale). This leads to the following functional form for the output supply and factor demand equations:

$$y = b \sum_{i=1}^{N_A+N_B} a_{ii} \left( \frac{w_i}{p} \right)^{\frac{1}{2}}$$

$$x_i^A = -b \sum_{j=1}^{N_A+N_B} a_{ij} \left( \frac{w_j}{w_i} \right)^{\frac{1}{2}} \quad i = 1, \cdots, N_A$$

$$x_i^B = -\frac{b \sum_{j=1}^{N_A+N_B} a_{ij} (w_j/w_i)^{\frac{1}{2}}}{1 + \sum_{j=1}^{N_A+N_B} \sum_{k=1}^{N_A+N_B} a_{jk} w_j^{-\frac{1}{2}} w_k^{\frac{1}{2}} + \sum_{j=1}^{N_A+N_B} a_{0j} p^{\frac{1}{2}} w_i^{\frac{1}{2}}} \quad i = N_A + 1, \cdots, N_A + N_B$$
(3.32)

Note that the particular functional form (3.31) implies $\partial \pi(w, p, b)/\partial b = \pi(w, p, b)/b$, so that the demand equations for inputs $x_B$ can be simplified to

$$x_i^B = -\left( \frac{b^2}{b + \pi} \right) \sum_{j=1}^{N_A+N_B} a_{ij} \left( \frac{w_j}{w_i} \right)^{\frac{1}{2}} \quad i = N_A + 1, \cdots, N_A + N_B$$
(3.33)

However, in the absence of constant returns to scale, the demand equations for inputs subject to a budget constraint generally will be nonlinear in the parameters to be estimated.

References

Chapter 4
Nonlinear Static Duality Theory (for a single agent)

4.1 The primal-dual characterization of optimizing behavior

In previous lecture we have constructed primal-dual relations such as

\[
G(w, p, x) \equiv \pi(w, p) - \{p f(x) - wx\} \quad \text{(page 33)} \quad (4.1a)
\]

\[
\tilde{G}(p, y, x) \equiv V(p, y) - u(x) \quad \text{((3.6) on page 30)} \quad (4.1b)
\]

\[
\tilde{G}(p, u, x) \equiv E(p, u) - p x
\]

where the first term denotes the optimal value of profits/utility/expenditure for an agent solving a profit maximization/utility maximization/cost minimization problem and the second term denotes a feasible level of profits/utility/expenditure, respectively. It is obvious that the primal-dual relations (4.1a)–(4.1b) attain a minimum value (equal to zero) at an equilibrium combination of choice variables \(x\) and parameters: \((w, p, x(w, p))\)/(\((p, y, x(p, y))\)). Likewise the primal-dual relations c for cost minimization attains a maximum value (equal to zero) at an equilibrium combination of choice variables \(x\) and parameters: \((p, u, x(p, u))\). This implies that, given equilibrium levels of \(x^*\) of the agent’s choice variables \(x\), the primal-dual relations (4.1a)–(4.1b) attain a minimum over possible values of the parameters at the particular level of the parameters for which \(x^*\) solves the profit/utility maximization problem:

\[
given \ x^0 \equiv x(w^0, p^0) \text{ solving a profit maximization problem (2.1):} \]

\[
(w^0, p^0) \text{ solves } \min_{w,p} G(w, p, x_0) \equiv \pi(w, p) - \{p f(x^0) - wx^0\} = 0 \quad (4.2)
\]

\[
given \ x^0 \equiv x(p^0, y^0) \text{ solving a utility maximization problem (3.1):}
\]
Nonlinear Static Duality Theory (for a single agent)

(p^0, y^0) solves \( \min_{p,y} \hat{G}(p, y, x^0) \equiv V(p, y) - u(x^0) = 0 \) s.t. \( px^0 = y \)

(4.3)

and similarly for case (4.1c), given \( x^0 = x(p^0, u^0) \) solving a cost minimization problem (3.2):

\( p^0 \) solves \( \max_p \hat{G}(p, u^0, x^0) \equiv E(p, u^0) - px^0 = 0 \)

(4.4)

where \( u^0 \equiv u(x^0) \). By analyzing the first order conditions for an interior solution to these problems, we easily derived Hotelling’s Lemma (page 12), Roy’s Theorem (page 29) and we can easily derive Shephard’s Lemma, respectively.

To repeat, it is obvious that the hypotheses of profit maximization, utility maximization and cost minimization imply that an equilibrium combination of choice variables and parameters are obtained by solving (4.2)–(4.4), respectively. In this sense a necessary condition for \( x^0 \) to solve a profit maximization problem (2.1) conditional on prices \((w^0, p^0)\) is that \((w^0, p^0)\) solve (4.2), and similarly for utility maximization and cost minimization.

A further question is: does \( x^0 \) solve a profit maximization problem (2.1) (e.g.) conditional on \((w^0, p^0)\) if and only if \((w^0, p^0)\) solve (4.2) conditional on \( x^0 \)? The answer is yes and this implies that problems (4.2)–(4.4) exhaust the implications of behavioral models (2.1), (3.1) and (3.2), respectively, for local comparative static analysis of changes in prices.

The intuitive explanation of this result is surprisingly simple (given the confusion that has been raised in the recent past over this matter, especially Silberberg 1974, pp. 159–72). For example, note that the profit maximizing derived demands \( x(w, p) \) solving \( \max_x \{ pf(x) - wx \} = \pi(w, p) \) also solve

\[ \min_x G(w, p, x) = \pi(w, p) - \{ pf(x) - wx \} = 0 \]

(4.5)

and note that any \( x \) such that \( G(w, p, x) = 0 \) also solves \( \max_x \{ pf(x) - wx \} \) conditional on \((w, p)\). Since the minimum value of \( G(w, p, x) \) over \((w, p)\) is also 0, it follows that any combination \((w^0, p^0, x^0)\) solving \( \min_{w,p} G(w, p, x^0) \) also solves \( \min_x G(w^0, p^0, x) \), and conversely any \((w^0, p^0, x^0)\) solving \( \min_x G(w^0, p^0, x) \) also solves \( \min_{w,p} G(w, p, x^0) \).

Thus solving \( \min_{w,p} G(w, p, x^0) \) is equivalent to solving the profit maximization problems \( \max_x \{ p^0 f(x) - w^0 x \} \). Also note that an (interior) global solution \((w^0, p^0)\) to \( \min_{w,p} G(w, p, x^0) \) implies for local comparative static purposes only that

\[ \frac{\partial G(w^0, p^0, x^0)}{\partial p} = 0, \quad \frac{\partial G(w^0, p^0, x^0)}{\partial w_i} = 0 \quad i = 1, \ldots, N \]

(4.6a)

\[ \frac{\partial^2 G(w^0, p^0, x^0)}{\partial p \partial w} = \text{symmetric positive semidefinite} \]

(4.6b)
4.2 Producer behavior

i.e. only the first and second order conditions for an interior minimum are relevant.\(^1\)

The implication of the above argument is that (4.6) exhaust the restrictions placed on the comparative static effects of (local) changes in prices \((w, p)\) by the hypothesis of competitive profit maximization for the individual firm. Similar conclusions hold for the cases of utility maximization (4.3) and cost minimization (4.4).

More generally, consider an optimization problem

\[
\max_x f(x, \alpha) \equiv \phi(\alpha)
\]

\[
\text{s.t. } g(x, \alpha) = 0
\]

(4.7)

where both the objective function \(f(x, \alpha)\) and constraint (or vector of constraint) \(g(x, \alpha) = 0\) are conditional on a vector of parameters \(\alpha\) (e.g. prices or parameters shifting the price schedules facing the agent). Then \(x^0\) solves (4.7) conditional on \(\alpha^0\) if and only if \(\alpha^0\) solves the following problem conditional on \(x^0\):

\[
\min_{\alpha} G(x^0, \alpha) \equiv \phi(\alpha) - f(x^0, \alpha)
\]

\[
\text{s.t. } g(x^0, \alpha) = 0
\]

(4.8)

Therefore the first and second order conditions for a solution to problem (4.8) in parameter \((\alpha)\) space exhaust the implications of the behavioral model (4.7) for comparative static effects of (local) changes in parameters \(\alpha\).\(^2\)

4.2 Producer behavior

Consider the following general profit maximization problem

\[
\max_x \{R(x, \alpha_A) - c(x, \alpha_B)\} \equiv \tilde{\pi}(x^*, \alpha) \equiv \pi(\alpha, b)
\]

\[
\text{s.t. } c(x, \alpha_C) = b
\]

(4.9)

where all prices may be endogenous to the producer and there is a budget constraint (or vector of constraints) \(c(x, \alpha_C) = b\) limiting expenditures on at least some inputs. Hence \(R(x, \alpha_A)\) denotes total revenue as a function of the output level \(y\) (or equivalent the inputs levels \(x\), given a single output production function \(y = f(x)\)) and parameters \(\alpha_A\) shifting the price schedules \(p(y, \alpha_A)\) facing the firm. \(c(x, \alpha_B)\) denotes total costs as a function of the input levels \(x\) and the parameters \(\alpha_B\) shifting

---

1 The condition \(G(w^0, p^0, x^0) = 0\) for a global solution to \(\min_{w, p} G(w, p, x^0)\) implies only that \(\pi(w^0, p^0) = p^0 f(x^0) - w^0 x^0\), which is not directly relevant to local comparative statics.

2 Likewise, if we are only interested in the comparative static effects of changes in a subset \(\alpha_B\) of parameters \(\alpha\), then \(x^0\) solves (4.7) conditional on \(\alpha^0\) if and only if \(\alpha^0_B\) solves \(\min_{\alpha_B} G(x^0, \alpha^0_A, \alpha^0_B)\) \(\text{s.t. } g(x^0, \alpha^0_A, \alpha^0_B) = 0\). In turn the first and second order conditions in the parameter \(\alpha_B\) subspace for this minimization problem exhaust the implications of (4.7) for the comparative static effects of changes in parameters \(\alpha_B\).
the factor supply schedules \( u_i(x, \alpha_B) \) (\( i = 1, \cdots, N \)) facing the firm. \( \pi(\alpha, b) \) denotes the dual profit function for (4.9), i.e. the relation between maximization attainable profits and parameters \( (\alpha, b) \). Formally (4.9) allows for traditional monopoly/monopsony behavior, the specification of financial constraints in terms of both fixed cash constraints (at level \( b \)) and costs of borrowing the vary with the level of borrowing (and the firm’s debt-equity ratio), the endogeneity of the opportunity cost (value of forgone leisure) of farm family labor, etc.

Consider the corresponding minimization problem

\[
\min_{\alpha, b} G(\alpha, b, x) \equiv \pi(\alpha, b) - \{R(x, \alpha_A) - c(x, \alpha_C)\} \\
\text{s.t. } c(x, \alpha_C) = b
\]

(4.10)

or equivalently (substituting out the budget constraint \( c(x, \alpha_C) = b \))

\[
\min_{\alpha} \tilde{G}(\alpha, x) \equiv \pi(\alpha, c(x, \alpha_C)) - \{R(x, \alpha_A) - c(x, \alpha_B)\} \\
\text{s.t. } c(x, \alpha_C) = b
\]

(4.11)

The analysis in the previous section implies that the following first and second order conditions for an (interior) solution to (4.11) exhaust the implications of profit maximization for the effects of local changes in parameters \( \alpha \): (using obvious vector notation):

\[
\hat{G}_\alpha(\alpha, x^*) = 0 \\
\hat{G}_{aa}(\alpha, x^*) \text{ symmetric positive semidefinite}
\]

(4.12)

where \( x^* = x(\alpha, b) \) solves (4.9), and

\[
\hat{G}_\alpha \equiv \begin{bmatrix} \frac{\partial \tilde{G}}{\partial \alpha} & \cdots & \frac{\partial \tilde{G}}{\partial \alpha} \end{bmatrix}_{1 \times Z}, \quad \hat{G}_{aa} \equiv \begin{bmatrix} \frac{\partial^2 \tilde{G}}{\partial \alpha \partial \alpha} \end{bmatrix}_{Z \times Z}
\]

Since \( \hat{G}(\alpha, x^*) \equiv \pi(\alpha, c(x^*, \alpha_C)) - \pi(x^*, \alpha) \), restrictions (4.12) can be rewritten as

\[
\hat{G}_\alpha(\alpha, x^*) \equiv \pi_\alpha(\alpha, b) + \pi_b(\alpha, b)c_\alpha(x^*, \alpha) - \pi_\alpha(x^*, \alpha) = 0 \quad (4.13a)
\]

\[
\hat{G}_{aa}(\alpha, x^*) \equiv \pi_{aa}(\alpha, b) + \pi_{ab}(\alpha, b)c_\alpha(x^*, \alpha) + \pi_{ba}(\alpha, b)c_\alpha(x^*, \alpha) + \pi_{bb}(\alpha, b)c_{aa}(x^*, \alpha) \\
+ \pi_{ab}(\alpha, b)c_\alpha(x^*, \alpha) - \pi_{aa}(x^*, \alpha) \quad \text{symmetric positive semidefinite} \quad (4.13b)
\]

In order to derive a system of estimating equations for a behavioral model of the type (4.9), we can begin by specifying a functional form \( \psi(\alpha, b) \) for the firm’s dual profit function \( \pi(\alpha, b) \). Then we calculate corresponding envelop relations (4.13a).

Note, however, that calculation of these envelop relations also requires knowledge of the derivatives \( c_\alpha(x, \alpha) \), \( R_\alpha(x, \alpha) \), \( C_\alpha(x, \alpha) \), i.e. we must specify functional forms for \( R(x, \alpha) \), \( C(x, \alpha) \) and \( c(x, \alpha) \) in addition to the functional form for \( \pi(\alpha, b) \). This does not appear to cause any serious problems: although these functions are not independent of the functional form for \( \pi(\alpha, b) \), a flexible specification (see next

\[\text{In the competitive case } R(x, \alpha_A) = pf(x), C(x, \alpha_B) = w x, c(x, \alpha_C) = w_C x_C; \text{ So that } \]

\[ R_{\alpha A} = y, C_{\alpha B} = x, c_{\alpha C} = x_C. \]
4.3 Consumer behavior

Consider the following general utility maximization problem

\[
\begin{align*}
\max_x u(x) & \equiv V(\alpha, y) \\
\text{s.t.} & \quad c(x, \alpha) = y
\end{align*}
\]

(4.15)

where these is a nonlinear budget constraint \( c(x, \alpha) = y \), i.e. in general prices of the commodities \( x \) can depend upon the levels of the commodities purchased by the consumer as well as upon the parameters \( \alpha \). Also consider the corresponding cost minimization problem

\[
\begin{align*}
\min_x c(x, \alpha) & \equiv E(\alpha, u) \\
\text{s.t.} & \quad u(x) = u
\end{align*}
\]

(4.16)

Comparative static effects for the maximization problem (4.15) can be analyzed in term of the primal-dual relation

\[
\begin{align*}
\min_{\alpha, y} G(\alpha, y, x) & \equiv V(\alpha, y) - u(x) \\
\text{s.t.} & \quad c(x, \alpha) = y
\end{align*}
\]

(4.17)

or equivalently (substituting out the budget constraint)

\[
\begin{align*}
\min_a \tilde{G}(\alpha, x) & \equiv V(\alpha, c(x, \alpha)) - u(x)
\end{align*}
\]

(4.18)

The first and second order conditions for a solution to (4.18) yield

\[
\tilde{G}_a(\alpha, x^*) \equiv V_a(\alpha, y) + V_y(\alpha, y)c_a(x^*, \alpha) = 0
\]

(4.19a)
\( \dot{G}_{\alpha \alpha}(x^*, \alpha) = V_{\alpha \alpha}(\alpha, y) + V_{\alpha y}(\alpha, y) c_\alpha(x^*, \alpha) + V_{y \alpha}(\alpha, y) c_\alpha(x^*, \alpha) + V_{y y}(\alpha, y) c_\alpha(x^*, \alpha) + V_{\alpha \alpha}(\alpha, y) \) symmetric positive semidefinite

Equation (4.19a) \( c_\alpha(x^*, \alpha) = -V_{\alpha \alpha}(\alpha, y)/V_{y}(\alpha, y) \) are a generalization of Roy’s Theorem.

Likewise comparative static effects for the minimization problem (4.16) can be analyzed in terms of the primal-dual relation

\[
\max \alpha H(\alpha, x) \equiv E(\alpha, u(x)) - c(x, \alpha)
\]

The first and second order conditions for a solution to (4.20) yield

\[
H_{\alpha}(\alpha, x^*) \equiv E_{\alpha}(\alpha, u^*) - c_\alpha(x^*, \alpha) = 0
\]

\[
H_{\alpha \alpha}(\alpha, x^*) \equiv E_{\alpha \alpha}(\alpha, u^*) - c_{\alpha \alpha}(x^*, \alpha)
\]

(4.21a) \( c_\alpha(x^*, \alpha) = E_{\alpha}(\alpha, u^*) \) is a generalization of Shephard’s Lemma.

In order to derive a generalization of the Slutsky equation, note that the Hicksonian demands \( x^h(\alpha, u) \) solving the cost minimization problem (4.16) for parameters \((\alpha, u)\) also solve the utility maximization problem (4.15) for parameters \((\alpha, b) = (\alpha, E(\alpha, u))\), i.e.

\[
x^h_i(\alpha, u) = x^M_i(\alpha, E(\alpha, u)) \quad i = 1, \ldots, N \quad \text{for all } \alpha
\]

Differentiating (4.22) with respect to \( \alpha \) (holding utility level \( u \) constant),

\[
\frac{\partial x^h_i(\alpha, u^*)}{\partial \alpha_j} = \frac{\partial x^M_i(\alpha, y)}{\partial \alpha_j} + \frac{\partial x^M_i(\alpha, y)}{\partial y} \frac{\partial E(\alpha, u^*)}{\partial \alpha_j} \quad i = 1, \ldots, N \quad j = 1, 2
\]

(4.23)

Substituting the generalized Shephard’s Lemma (4.21a) into (4.23),

\[
\frac{\partial x^M_i(\alpha, y)}{\partial \alpha_j} = \frac{\partial x^h_i(\alpha, u^*)}{\partial \alpha_j} - \frac{\partial x^M_i(\alpha, y)}{\partial y} \frac{\partial c(x^*, \alpha)}{\partial \alpha_j} \quad i = 1, \ldots, N \quad j = 1, 2
\]

(4.24)

where \( \partial c(x^*, \alpha)/\partial \alpha_j \neq x_j(\alpha, y) \) except in the competitive case, where \( c(x, \alpha) \equiv \sum_{i=1}^{N} \alpha_i x_i (\alpha_i \equiv w_i) \). Also note that the second order conditions for cost minimization (4.21b) do not reduce to \( \frac{\partial c(x^*, \alpha)}{\partial \alpha_j} \) symmetric negative semidefinite except in the competitive case.

An alternative generalization of the Slutsky equation that directly relates Marshallian demands to the second order conditions for cost minimization (4.21b) can be derived as follows. The first order conditions for cost minimization \( E_{\alpha}(\alpha, u^*) - c_\alpha(x^*, \alpha) = 0 \) (4.21a) can be rewritten as

\[
E_{\alpha}(\alpha, u^*) = c_\alpha(x^*, \alpha) \quad \text{for all } \alpha \quad \text{at } x^* \equiv x^M(\alpha, E(\alpha, u^*))
\]

(4.25)
using (4.22). Differentiating (4.25) with respect to \( \alpha \),
\[
E_{aa}(\alpha, u^*) = c_{aa}(x^*, \alpha) + c_{ax}(x^*, \alpha) \left[ x^M_\alpha(\alpha, y) + x^M_y(\alpha, y)E_{a}(\alpha, u^*) \right]
\] (4.26)

substituting (4.25) into (4.26) and rearranging,
\[
c_{a,x}(x^*, \alpha) \left[ x^M_\alpha(\alpha, y) + x^M_y(\alpha, y) c_a(x^*, \alpha) \right] = E_{aa}(\alpha, u^*) - c_{aa}(x^*, \alpha)
\]

symmetric negative semidefinite
(4.27)

by the second order conditions (4.21b) for cost minimization.

In order to derive a system of estimating equations for a behavioral model of the type (4.15), we can begin by specifying a functional form for the consumer’s indirect utility function \( V.\alpha; y/ \) and the budget constraint \( c(x, \alpha) = y \). Then we calculate corresponding envelope relations \( c_a(x^*, \alpha) = -V_y(\alpha, y)/V_y(\alpha, y) \) (4.19a) which we then solve for consumer demand relations \( x^* = x(\alpha, y) \). Since utility maximization and cost minimization are equivalent, the Marshallian demand equations integrate up to a macrofunction \( V(\alpha, y) \) representing utility maximization behavioral if and only if the corresponding envelope relations \( c_a(x^*, \alpha) = E_a(\alpha, u^*) \) for cost minimization satisfy the symmetry and second order conditions (4.21b) for integration up to a macrofunction \( E(\alpha, u^*) \) representing cost minimizing behavior. These symmetry and second order conditions are related to the parameters of the estimated Marshallian demand equations using the generalized Slutsky equations (4.27).

**References**

Chapter 5
Functional Forms for Static Optimizing Models

5.1 Difficulties with simple linear and log-linear Models

The main purpose of this lecture is to discuss the concept of flexible functional forms and to present specific functional forms that are commonly employed with static duality theory. In order to appreciate the potential value of such functional forms, we begin with a discussion of the problem in using simpler linear and log-linear models. For simplicity we restrict this discussion to cost-minimizing behavior.

First suppose that a simple linear model of a cost function \( c(w, y) \) is formulated:

\[
c = a_0 + \sum_{i=1}^{N} a_i w_i + a_y y. \tag{5.1}
\]

If a producer minimizes his total cost of production, then Shephard’s Lemma applies to the cost function. Applying Shephard’s Lemma to (5.1),

\[
x_i = \frac{\partial c(w, y)}{\partial w_i} = a_i \quad i = 1, \cdots, N. \tag{5.2}
\]

i.e. the cost-minimizing factor demands \( x = x(w, y) \) are in fact independent of factor prices and the level of output. Alternatively suppose that factor demands are estimated as a linear function of prices and output:

\[
x_i = a_{i0} + \sum_{j=1}^{N} a_{ij} w_j + a_{iy} y_i = 1, \cdots, N. \tag{5.3}
\]

However if factor demands are homogeneous of degree 0 in prices then (by Euler’s theorem) \( \sum_{j=1}^{N} \frac{\partial x_i(w, y)}{\partial w_j} w_j = 0 \quad (i = 1, \cdots, N.) \) (see footnote 2 on page 3). So (5.3) is consistent with cost minimizing behavior only if (5.3) reduces to

\[
x_i = a_{i0} + a_{iy} y \quad i = 1, \cdots, N. \tag{5.4}
\]
i.e. cost minimizing factor demands are independent of factor prices (implying a Leontief production function).

Of course these difficulties can be circumvented by first normalizing prices on a numeraire price:

\[
x_i = a_{i0} + \sum_{j=2}^{N} a_{ij} \left( \frac{w_j}{w_1} \right) + a_{iy} y \quad i = 1, \cdots, N.
\]

(5.5)

Here the hypothesis that factor demands are homogenous of degree 0 in prices is imposed a priori by linearizing on normalized prices; so homogeneity cannot place any further restrictions on the functional form (5.5) for factor demands. Thus, to the extent that factor demands are homogenous of degree 0 in prices, (5.5) clearly is a better specification of factor demands than is (5.3). Nevertheless, note that (5.5) does impose an symmetry on the effects of the numeraire price \( w_1 \) relative to the effects of other prices on factor demands. Responses \( \frac{\partial c}{\partial w_j} \) where \( j \neq 1 \) can be calculated directly from (5.5) are simply \( \frac{\partial c}{\partial w_j} = a_{ij} \), but responses \( \frac{\partial c}{\partial w_1} \) must be calculated indirectly from the homogeneity condition \( \frac{\partial c}{\partial w_1} w_1 + \sum_{j=2}^{N} \frac{\partial c}{\partial w_j} w_j = 0 \) as \( \frac{\partial c}{\partial w_1} = -\sum_{j=2}^{N} a_{ij} \left( \frac{w_j}{w_1} \right) \) \( (i = 1, \cdots, N) \).

Next consider simple log-linear models of cost minimizing behavior. A log-linear cost function (corresponding to a cobb-Douglas technology)

\[
\ln c = a_0 + \sum_{i=1}^{N} a_i \ln w_i + a_y \ln y
\]

(5.6)

implies that \( \frac{\partial \ln c}{\partial \ln w_i} = a_i \), but \( \frac{\partial \ln c}{\partial \ln w_j} = \frac{\partial c}{\partial w_i} / c = \frac{w_i x_i}{c} \) using Shephard’s Lemma. Then the share of each input in that costs is independent of prices and output:

\[
s_i = \frac{w_i x_i}{c} = a_i \quad i = 1, \cdots, N.
\]

(5.7)

Alternatively consider log-linear factor demands:

\[
\ln x_j = a_{j0} + \sum_{i=1}^{N} a_{ij} \ln w_j + a_{iy} y \quad i = 1, \cdots, N.
\]

(5.8)

Homogeneity implies \( \sum_{j=1}^{N} \frac{\partial x_j}{\partial w_j} / \frac{1}{w_j} = 0 \) \((i = 1, \cdots, N)\) and \( \frac{\partial \ln x_j}{\partial \ln w_j} = \frac{\partial x_j}{\partial w_j} / \frac{x_j}{w_j} \) using Shephard’s Lemma; so homogeneity of factor demands does not imply the restriction \( \sum_{j=1}^{N} a_{ij} = 0 \) \((i = 1, \cdots, N)\). On the other hand, in the case of log-linear consumer demands \( x = x(p, y) \) conditional on prices \( p \) and income \( y \), the adding up constraint \( \sum_{i=1}^{N} \frac{\partial x_i(p, y)}{\partial y} p_i = 1 \) (derived by differentiating the budget constraint \( px = y \) with respect to \( y \)) is generally satisfied only if all income elasticities are equal to 1 (see Deaton and Muellbauer 1980, pp. 16–17).
5.2 Second order flexible functional forms

The previous section illustrated the severe restrictions implied by linear or log-linear models of cost functions or (by extension) profit functions or indirect utility functions. These functional forms imply extremely restrictive production functions and behavioral relations. Likewise the most commonly employed production functions, i.e. Cobb-Douglas or CES, place significant restrictions on behavioral relations (all elasticities of substitution equal 1 or all elasticities of substitutions are equal, respectively). Also note that a Cobb-Douglas production function is equivalent to a Cobb-Douglas functional form for the associated cost function and profit function (e.g. Varian 1984, p. 67).

The concept of second order flexible functional forms has been employed in order to generate less restrictive functional forms for behavioral models (see Diewert 1971, pp. 481–507).

Suppose that the true production function, cost function, profit function or indirect utility function for an agent is represented by the functional form $f(x)$. Then, taking a second order Taylor series approximation of $f(x)$ about a point $x_0$,

$$f(x_0 + \Delta x) \approx f(x_0) + \sum_i \frac{\partial f(x_0)}{\partial x_i} \Delta x_i + \frac{1}{2} \sum_i \sum_j \frac{\partial^2 f(x_0)}{\partial x_i \partial x_j} \Delta x_i \Delta x_j. \quad (5.9)$$

Thus for a small $\Delta x$, $f(x_0 + \Delta x)$ can be closely approximated in terms of the level of $f$ at $x_0$ ($f(x_0)$) and its first and second derivatives at $x_0$ ($\frac{\partial f(x_0)}{\partial x_i}, \frac{\partial^2 f(x_0)}{\partial x_i \partial x_j}$).

In turn, any other function $g(x)$ whose level at $x_0$ can equal the level of $f$ at $x_0$ ($g(x_0) = f(x_0)$) and whose first and second derivatives at $x_0$ can equal the first and second derivatives of $f$ at $x_0$ ($\frac{\partial g(x_0)}{\partial x_i} = \frac{\partial f(x_0)}{\partial x_i}, \frac{\partial^2 g(x_0)}{\partial x_i \partial x_j} = \frac{\partial^2 f(x_0)}{\partial x_i \partial x_j}$) can closely approximate $f(x)$ for small variations in $x$ about $x_0$.

To be more formal, $g(x)$ provides a second order (differential) approximation to $f(x)$ at $x_0$ if and only if

$$g(x_0) = f(x_0) \quad (5.10a)$$

$$\frac{\partial g(x_0)}{\partial x_i} = \frac{\partial f(x_0)}{\partial x_i} \quad i = 1, \ldots, N. \quad (5.10b)$$

$$\frac{\partial^2 g(x_0)}{\partial x_i \partial x_j} = \frac{\partial^2 f(x_0)}{\partial x_i \partial x_j} \quad i, j = 1, \ldots, N. \quad (5.10c)$$

In general the true functional form $f(x)$ is unknown. Thus $g(x)$ provides a second order flexible approximation to an arbitrary function $f(x)$ at $x_0$ if conditions (5.10) can be satisfied at $x_0$ for any function $f(x)$. In other words, $g(x)$ is a second order flexible functional form if, at any point $x_0$, any combination of level $g(x_0)$ and derivatives $\frac{\partial g(x_0)}{\partial x_i}, \frac{\partial^2 g(x_0)}{\partial x_i \partial x_j}$ can be attained. Thus a general second order flexible functional form $g(x)$ must have at least $1 + N + \frac{N(N+1)}{2}$ free parameters (assuming $\frac{\partial^2 g}{\partial x_i \partial x_j}$ symmetric). If $g(x)$ is linear homogeneous in $x$, then (using Euler’s theorem)
there are \(1 + N\) restrictions.

\[
g(x_0) = \sum_{i=1}^{N} \frac{\partial g(x_0)}{\partial x_i} x_{i0}
\]

\[
0 = \sum_{j=1}^{N} \frac{\partial^2 g(x_0)}{\partial x_i \partial x_j} x_{j0} \quad i = 1, \ldots, N
\]

on \(g(x_0)\) and its first and second derivatives. Then a linear homogeneous second order flexible functional form \(g(x)\) has \(\frac{N(N+1)}{2}\) free parameters. Note that the number of free parameters \(\frac{N(N+1)}{2}\) increase exponentially with the dimension \(N\) of \(x\).

Consider a functional form \(c(w, y)\) for a producer’s cost function. \(c(w, y)\) is a second order flexible functional form if \(c(0_0, y_0), \frac{\partial c(0_0, y_0)}{\partial w}, \frac{\partial c(0_0, y_0)}{\partial y}, \frac{\partial^2 c(0_0, y_0)}{\partial w \partial y}\) are not restricted a priori (except for homogeneity restrictions (5.11)) at \((0_0, y_0)\). Moreover a second order flexible approximation \(c(w, y)\) to a true cost function implies a second order flexible approximation to a true production function (see equation (1.5), (1.6) on page (1.5)).

Next consider a functional form \(\pi(w, p)\) for a firm’s profit function. \(\pi(w, p)\) is a second order flexible functional form if the combination \(\pi(0_0, 0_0), \frac{\partial \pi(0_0, 0_0)}{\partial w}, \frac{\partial \pi(0_0, 0_0)}{\partial p}, \frac{\partial^2 \pi(0_0, 0_0)}{\partial w \partial p}\) is not restricted a priori (except for homogeneity restrictions). In addition a second order flexible approximation to a true profit function implies a second order flexible approximation to a true production function (see equation (2.5), (2.6) of page (2.5)). Likewise a second order approximation \(V(p, y)\) to a true indirect utility function implies a second order approximation to a true utility function (structure of preferences).

Thus, to the extent that \(\Delta w, \Delta y\) is small over the data set \((w, y)\), a second order flexible functional form for a producer’s cost function \(c(w, y)\) provides a close approximation to the true cost function and to the underlying production function. Similar comments apply to profit functions and indirect utility functions. Note that if prices \(w\) are highly co-linear over time then in effect the variation \(\Delta w\) in prices may be small. Thus second order flexible functional forms may be useful in modeling behavior over data sets with very high multicollinearity. On the other hand, for a cost function, changes in output \(\Delta y\) are likely to be substantial over time and not perfectly correlated with changes in factor prices \(\Delta w\). In this case it may be desirable to provide a third order or even higher order of approximation in output \(y\) to the true cost function, e.g. by specifying a cost function \(c(w, y)\) such that any combination of the following cost and derivatives can be attained at a point \((0_0, 0_0)\) (subject to homogeneity restrictions):
Notice, however, that if third derivatives (5.12d) as well as (5.12a)–(5.12c) to be free then a greater number of free parameters must be estimated in the model, which implies a lost of degrees of freedom in the estimation.

In sum, there are two serious problems in the application of flexible functional forms. First, the number of parameters to be estimated increases exponentially with the number of variations (e.g. prices, outputs) included in the functional form and with the order of the Taylor series approximation. Thus flexible functional forms generally require a high level of aggregation of commodities; but consistent aggregation of commodities is possible only under strong restrictions on the underlying technology or preference structure (see next lecture). Second, when there is substantial variation in the data, the global properties of flexible functional forms (how well these forms approximate the unknown true function \( f(x) \) over large variation in prices and output) become very important. Unfortunately the global properties of many common flexible functional forms are not clear. It is often difficult to discern whether these functional forms impose restrictions over large \( \Delta x \) that are unreasonable on a priori grounds and hence seriously bias econometric estimates (see pages 232–236 of M. Fuss, D. McFadden, Y. Mundlak, “A Survey of Functional Forms in the Economic Analysis of Production,” in M. Fuss, D. McFadden, Production Economic: A Dual Approach to Theory and Applications, 1978 for a good early discussion of this problem). Nevertheless the concept of flexible functional forms appears to be useful in practice.

5.3 Examples of second order flexible functional forms

The most common flexible functional forms are the Translog and Generalized Leon-tief, so we focus on these plus the Normalized Quadratic (which is the most obvious flexible functional form). First consider dual profit functions \( \pi(w, p) \), which we now write as \( \pi(v) \) where \( v \equiv (w, p) \).

The most obvious candidate for a flexible functional form for a dual profit function \( \pi(v) \) is the quadratic:

\[
\pi(v) = a_0 + \sum_{i=1}^{N} a_i v_i + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} v_i v_j \tag{5.13}
\]
Note that this quadratic can be viewed as a second order expansion of \( \pi \) in powers of \( v \). In the absence of homogeneity restrictions, (5.13) obviously is a second order flexible functional form: differentiating twice yields \( \frac{\partial^2 \pi}{\partial v_i \partial v_j} = a_{ij} \) \((i, j = 1, \ldots, N)\) i.e. each second derivative is determined as a free parameter \( a_{ij} \); differentiating once yields \( \frac{\partial \pi}{\partial v_i} = a_i + \sum_{j=1}^{N} a_{ij} v_j \) \((i = 1, \ldots, N)\) i.e. there is a remaining free parameter \( a_i \) to determine \( \frac{\partial \pi}{\partial v_i} \) at any level; and finally the parameter \( a_0 \) is free to determine \( \pi \) at any level. However linear homogeneity of \( \pi(v) \) in \( v \) implies (by Euler’s theorem) \( \sum_{j=1}^{N} \frac{\partial^2 \pi}{\partial v_i \partial v_j} v_j = 0 \) \((i = 1, \ldots, N)\) and in turn (by Hotelling’s Lemma) \( \sum_{j=1}^{N} a_{ij} v_j = 0 \). Thus the quadratic profit function (5.13) reduces to a linear profit function:

\[
\pi = a_0 + \sum_{i=1}^{N} a_i v_i \tag{5.14}
\]

which implies (using Hotelling’s Lemma) that output supplies and factor demands are independent of prices \( v \).

This problem is circumvented by defining the quadratic profit function in terms of normalized prices:

\[
\bar{\pi}(\bar{v}) = a_0 + \sum_{i=1}^{N} a_i \bar{v}_i + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} \bar{v}_i \bar{v}_j \tag{5.15}
\]

where \( \bar{v}_i \equiv v_i / v_0 \) \((i = 1, \ldots, N)\), i.e. the inputs and outputs of the firm are indexed \( i = 0, \ldots, N \) and \( v_0 \) is chosen as the numeraire. By construction (5.15) satisfies the homogeneity condition (i.e. only relative prices \( \bar{v} \) matter) so homogeneity does not place any further restrictions on the functional form (5.15). Since \( \bar{v}_i \equiv v_i / v_0 \) implies \( d \bar{v}_i = \frac{1}{v_0} d v_i \) (for \( v_0 \) fixed), the derivatives of \( \bar{\pi}(\bar{v}) \) and the corresponding \( \pi(v) \) can be related simply as follows, \( \frac{\partial \bar{\pi}(\bar{v})}{\partial \bar{v}_i} = \frac{\partial \pi(v)}{\partial v_i}, \frac{\partial^2 \bar{\pi}(\bar{v})}{\partial \bar{v}_i \partial \bar{v}_j} = \frac{1}{v_0} \frac{\partial^2 \pi(v)}{\partial v_i \partial v_j} \) \((i, j = 1, \ldots, N)\). The derivatives of \( \pi(v) \) with respect to \( v_0 \) can then be recovered from (5.15) using the homogeneity conditions (5.11). Often calculating the data \( \bar{v} \) from the data \( (v_0, v_i, \ldots, v_N) \) we can assume without loss of generality (since only relative prices matter) that \( v_0 \equiv 1 \) and specify the estimating equations as

\[
y_i = \frac{\partial \bar{\pi}(\bar{v})}{\partial \bar{v}_i} = a_i + \sum_{j=1}^{N} a_{ij} \bar{v}_j \tag{5.16}
\]

\[
x_i = -\frac{\partial \bar{\pi}(\bar{v})}{\partial \bar{v}_j} = -a_i - \sum_{j=1}^{N} a_{ij} \bar{v}_j \quad i = 1, \ldots, N
\]

for outputs \( y \) and inputs \( x \). The corresponding equation for the numeraire commodity can be recovered from (5.16) using homogeneity. These equations (5.16) integrate up to a macrofunction \( \bar{\pi}(\bar{v}) \) if \( \frac{\partial^2 \bar{\pi}(\bar{v})}{\partial \bar{v}_i \partial \bar{v}_j} \) is symmetric, i.e. if \( a_{ij} = a_{ji} \).
(i, j = 1, · · · , N), and profit maximization implies that the matrix \( \frac{\partial^2 \pi(v)}{\partial v_i \partial v_j} = [a_{ij}] \) is symmetric positive semidefinite.

Next consider a Generalized Leontief dual profit function:

\[
\pi(v) = a_0 + \sum_{i=1}^{N} a_i \sqrt{v_i} + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} \sqrt{v_i} \sqrt{v_j} \tag{5.17}
\]

This is a second order expansion of \( \pi \) in powers of \( \sqrt{v} \). Note that \( \sum_j a_{ij} \sqrt{v_j} \sqrt{v_j} = \lambda \sum_j a_{ij} \sqrt{\lambda v_i} \sqrt{\lambda v_j} \) whereas \( \sum_i a_i \sqrt{\lambda v_i} = \sqrt{\lambda} \sum_i a_i \sqrt{v_i} \); so \( \pi(\lambda v) = \lambda \pi(v) \) requires \( a_0 = 0, a_i = 0 \) (i = 1, · · · , N). Thus, imposing the restriction of linear homogeneity, the Generalized Leontief \( \pi(v) \) can be rewritten as

\[
\pi(v) = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} \sqrt{v_i} \sqrt{v_j} \tag{5.18}
\]

Differentiating \( \pi(v) \) (5.18) once yields \( \frac{\partial \pi(v)}{\partial v_i} = a_{ii} + \sum_{j \neq i} a_{ij} \left( \frac{v_j}{v_i} \right)^{1/2} \) (i = 1, · · · , N), and differentiating \( \pi(v) \) twice yields

\[
\begin{align*}
\frac{\partial^2 \pi(v)}{\partial v_i \partial v_j} &= \frac{a_{ij}}{\sqrt{v_i} \sqrt{v_j}} \quad j \neq i \\
\frac{\partial^2 \pi(v)}{\partial v_i \partial v_i} &= -\sum_{j \neq i} a_{ij} \frac{\sqrt{v_j}}{v_i^{3/2}} \quad i, j = 1, \cdots, N \tag{5.19}
\end{align*}
\]

This provides a second order flexible form for \( \pi(v) \) subject to homogeneity restrictions. The corresponding estimating equations are

\[
\begin{align*}
y_i &= \frac{\partial \pi(v)}{\partial v_i} = a_{ii} + \sum_{j \neq i} a_{ij} \left( \frac{v_j}{v_i} \right)^{1/2} \\
x_i &= -\frac{\partial \pi(v)}{\partial v_i} = -a_{ii} - \sum_{j \neq i} a_{ij} \left( \frac{v_j}{v_i} \right)^{1/2} \quad i = 1, \cdots, N \tag{5.20}
\end{align*}
\]

for outputs \( y \) and inputs \( x \). These equations integrate up to a macrofunction \( \pi(v) \) if \( \frac{\partial^2 \pi(v)}{\partial v_i \partial v_j} \) is symmetric, i.e. \( a_{ij} = a_{ji} \) (i, j = 1, · · · , N) (see (5.19)). Profit maximization implies that the matrix \( \frac{\partial^2 \pi(v)}{\partial v_i \partial v_j} \) is symmetric positive semidefinite.

Third consider a Translog dual profit function

\[
\ln \pi(v) = a_0 + \sum_{i=1}^{N} a_i \ln v_i + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} \ln v_i \ln v_j \tag{5.21}
\]
This is a second order expansion of $\ln \pi$ in powers of $\ln v$. In order to determine homogeneity restrictions, note that $\pi(\lambda v) = \lambda \pi(v)$ implies $\ln \pi(\lambda v) = \ln \lambda + \ln \pi(v)$. Multiplying prices $v$ by $\lambda$ in (5.21).

\[
\ln \pi(\lambda v) = a_0 + \sum_i a_i \ln(\lambda v_i) + \sum_i \sum_j a_{ij} \ln(\lambda v_i) \ln(\lambda v_j) \\
= a_0 + \ln \lambda \sum_i a_i + \sum_i a_i \ln v_i + (\ln \lambda)^2 \sum_i \sum_j a_{ij} \\
+ 2 \ln \lambda \sum_i \sum_j a_{ij} \ln v_i + \sum_i \sum_j a_{ij} \ln v_i \ln v_j \\
= \ln \pi(v) + \ln \lambda \sum_i a_i + 2 \ln \lambda \sum_i \left( \sum_j a_{ij} \right) \ln v_i + (\ln \lambda)^2 \sum_i \sum_j a_{ij}
\]

(5.22)

Thus the linear homogeneity condition $\ln \pi(\lambda v) = \ln \lambda + \ln \pi(v)$ is satisfied if

\[
\sum_{i=1}^N a_i = 1 \tag{5.23}
\]

\[
\sum_{j=1}^N a_{ij} = 0 \quad i = 1, \ldots, N
\]

Rearranging $\frac{\partial \ln \pi}{\partial \ln v_j} = \frac{\partial \pi}{\partial v_j} / \pi$ as $\frac{\partial \pi}{\partial v_j} = \frac{\partial \ln \pi}{\partial \ln v_j} \cdot \pi / v_j$ and differentiating with respect to $v_j$,

\[
\frac{\partial^2 \pi(v)}{\partial v_i \partial v_j} = \frac{\partial^2 \ln \pi}{\partial \ln v_i \partial \ln v_j} \cdot \frac{\pi}{v_i} + \frac{\partial \ln \pi}{\partial \ln v_i} \frac{\partial \pi}{\partial v_j} \cdot \frac{1}{v_j} \\
= \frac{\pi}{v_i v_j} \left[ a_{ij} + \frac{\partial \ln \pi}{\partial \ln v_i} \frac{\partial \ln \pi}{\partial \ln v_j} \right] \quad i \neq j
\]

(5.24)

\[
\frac{\partial^2 \pi(v)}{\partial v_i \partial v_i} = \frac{\partial^2 \ln \pi}{\partial \ln v_i \partial \ln v_i} \cdot \frac{\pi}{v_i} + \frac{\partial \ln \pi}{\partial \ln v_i} \frac{\partial \pi}{\partial v_i} \cdot \frac{1}{v_i} - \frac{\partial \ln \pi}{\partial \ln v_i} \left( \frac{\partial \ln \pi}{\partial \ln v_i} \right)^2
\]

\[
= \frac{\pi}{(v_i)^2} \left[ a_{ii} + \left( \frac{\partial \ln \pi}{\partial \ln v_i} \right)^2 - \frac{\partial \ln \pi}{\partial \ln v_i} \right] \quad i, j = 1, \ldots, N
\]

since $\frac{\partial^2 \ln \pi}{\partial \ln v_i \partial \ln v_j} = a_{ij}$ (5.20), $\frac{\partial \ln v_j}{\partial v_j} = \frac{1}{v_j}$ and $\frac{\partial \pi}{\partial v_j} = \frac{\partial \ln \pi}{\partial \ln v_j} \cdot \pi / v_j$. Thus, as in the cost of the normalized Quadratic and Generalized Leontief, the differential equations $y_i(v) = \frac{\partial \pi(v)}{\partial v_i}$ (for outputs), $x_i(v) = -\frac{\partial \pi(v)}{\partial v_i}$ (for inputs) integrate up to a macrofunction $\pi(v)$ if $a_{ij} = a_{ji}$ ($i, j = 1, \ldots, N$). Profit maximization further requires that the matrix $\frac{\partial^2 \pi(v)}{\partial \ln v_i \partial \ln v_j}$ defined by (5.24) is symmetric positive semidefinite.

In contrast to the normalized Quadratic and Generalized Leontief, the Translog model (5.21) is more easily estimated in terms of share equations rather than output supply and factor demand equations per se. The elasticity formula $\frac{\partial \ln \pi}{\partial \ln v_j} = \frac{\partial \pi}{\partial v_j} / \pi$.
and Hotelling’s Lemma yield $y_i = \frac{\partial \ln \pi}{\partial \ln v_j}$, $x_i = -\frac{\partial \ln \pi}{\partial \ln v_j}$; so substituting for $\frac{\partial \ln \pi}{\partial \ln v_j}$ from (5.21) yields estimating equations for $y$ and $x$ that are nonlinear in the parameters $(a_0, \ldots, a_N, a_{11}, \ldots, a_{NN})$ which are to be estimated. On the other hand, the elasticity formula and Hotelling’s Lemma directly imply $p_i y_i = \frac{\partial \ln v_i}{\partial \ln v_j}$ and $w_i x_i = -\frac{\partial \ln v_i}{\partial \ln v_j}$. Thus the Translog model (5.21) can be most easily estimated in terms of equations for “profit shares”:

\[
\begin{align*}
  s_i &= \frac{p_i y_i}{\pi} = a_i + \sum_{j=1}^{N} a_{ij} \ln v_j \\
  s_i &= \frac{w_i x_i}{\pi} = -a_i - \sum_{j=1}^{N} a_{ij} \ln v_j \quad i = 1, \ldots, N
\end{align*}
\]

The homogeneity condition (5.23) and reciprocity conditions $a_{ij} = a_{ji}$ ($i, j = 1, \ldots, N$) can easily be tested, and the restriction $\frac{\partial^2 \pi(v)}{\partial v_i \partial v_j}$ positive semidefinite can be checked at data points $(v, \pi)$ using (5.24).

The only complication in this procedure for estimating Translog models arises from the fact that the sum of the output shares minus the sum of the input shares equals $1: \sum_i p_i y_i - \sum_i w_i x_i = \frac{\pi}{\pi} = 1$. Writing these equations as $s_i = \pm (a_i + \sum_{j=1}^{N} a_{ij} \ln v_j) + e_i$ ($i = 1, \ldots, N$) where $e_i$ denotes the disturbance for equation $i$, we see that this linear dependence between shares implies a linear dependence between disturbances, i.e. $e_i$ for any equation must be a linear combination of the disturbances for all other equations. This in turn implies that one of the share equations must be dropped from the econometric model for the purposes of estimation, but this is not a serious problem since there are simple techniques for insuring that econometric results are invariant with respect to which equation is actually dropped.

The above functional forms are easily generalized to cost functions $c(w, y)$, and the homogeneity, reciprocity and concavity conditions are calculated in an analogous manner. The normalized quadratic cost function can be written as

\[
\hat{c}(\hat{w}, y) = a_0 + \sum_{i=1}^{N} a_i \hat{w}_i + a_{yy} y^2 + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} \hat{w}_i \hat{w}_j + \sum_{i=1}^{N} a_{iy} \hat{w}_i y + a_{yy} y^2
\]

(5.26)

where $\hat{w}_i \equiv \frac{w_i}{w_0}$, and the corresponding factor demand equations are (using Shepard’s Lemma)

\[
\begin{align*}
  x_i &= \frac{\partial \hat{c}(\hat{w}, y)}{\partial \hat{w}_i} \\
  &= a_i + \sum_{j=1}^{N} a_{ij} \hat{w}_j + a_{iy} y \quad i = 1, \ldots, N
\end{align*}
\]

(5.27)
The effects of changes in the numeraire price \( w_0 \) can be calculated from (5.27) using the homogeneity restrictions (5.11). Note that the parameters \( a_y \) and \( a_{y^2} \) are not included in the factor demand equations, so that \( \partial c(w, y) / \partial y \) cannot be recovered directly from (5.27). Nevertheless \( \partial c(w, y) / \partial y \) can be calculated indirectly from (5.27) using the homogeneity condition \( \partial c(\lambda w, \lambda y) / \partial y = \lambda \partial c(w, y) / \partial y \) and the reciprocity relations \( \partial^2 c(w, y) / \partial w_i \partial y = \partial^2 c(w, y) / \partial y \partial w_i \) (i = 1, \cdots, N) plus Shephard’s Lemma (See Footnote 2 on page 19). Alternatively the cost function (5.26) can be estimated directly along with \( N - 1 \) of the factor demand equations (5.27).

The Generalized Leontief cost function is often written as

\[
e(w, y) = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} \sqrt{w_i} \sqrt{w_j} + y^2 \sum_{i=1}^{N} a_{iy} w_i
\]  

(5.28)

where \( a_{iy} = 0 \) i = 1, \cdots, N) implies that the production function is constant returns to scale. The corresponding factor demand equations are

\[
x_i = \frac{\partial c(w, y)}{\partial w_i} = a_{ii} y + \sum_{j \neq i} a_{ij} y \left( \frac{w_j}{w_i} \right)^{\frac{1}{2}} + a_{iy} y^2 \quad i = 1, \cdots, N \]  

(5.29)

The Translog cost function is

\[
\ln c(w, y) = a_0 + \sum_{i=1}^{N} a_i \ln w_i + a_y \ln y + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} \ln w_i \ln w_j \\
+ \sum_{i=1}^{N} a_{iy} \ln w_i \ln y + a_{y^2} \ln y^2
\]  

(5.30)

and the corresponding factor share equations are

\[
s_i \equiv \frac{w_i x_i}{c} = a_i + \sum_{j=1}^{N} a_{ij} \ln w_j + a_{iy} \ln y \quad i = 1, \cdots, N
\]  

(5.31)

As in the case of the Translog profit function, \( \sum_{i=1}^{N} s_i = 1 \), so that one share equation must be deleted from the estimation.

Finally, we briefly consider indirect utility functions \( V(p, y) \) where \( p \) denotes prices for consumer goods and \( y \) now denotes consumer income. The Normalized Quadratic indirect utility function can be written as

\[
\tilde{V}(\tilde{p}) = a_0 + \sum_{i=1}^{N} a_i \tilde{p}_i + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} \tilde{p}_i \tilde{p}_j
\]  

(5.32)
where $\tilde{p}_i \equiv p_i / y$, so that (5.32) is by construction homogeneity of degree 0 in $(p, y)$. $\partial V(p, y) / \partial y$ can be calculated from (5.32) using the condition $\sum_{i=1}^{N} \frac{\partial V(p, y)}{\partial p_i} p_i + \frac{\partial V(p, y)}{\partial y} y = 0$ implied by zero homogeneity. Then the consumer demand equations can be calculated using Roy’s identity $x_i = \frac{\partial V(p, y)}{\partial p_i} = \frac{\partial V(p, y)}{\partial y}$. Roy’s identity generally leads to equations that are nonlinear in the parameters $(\alpha)$ to be estimated. A simple example of a Translog indirect utility function is

$$\ln V(p) = a_0 + \sum_{i=1}^{N} a_i \ln \tilde{p}_i + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} \ln \tilde{p}_i \ln \tilde{p}_j$$  (5.33)

where again $\tilde{p}_i \equiv p_i / y$ ($i = 1, \ldots, N$). (see Varian 1984, pp. 184–186 for further examples of functional forms for indirect utility functions).

### 5.4 Almost ideal demand system (AIDS)

Suppose consumer expenditure function in the form

$$\log E(p, u) = a(p) + b(p) u$$  (5.34)

where

$$a(p) = a_0 + \sum_{i=1}^{M} \alpha_i \log p_i + \frac{1}{2} \sum_{i=1}^{M} \sum_{j=1}^{M} r_{ij}^* \log p_i \log p_j$$

$$b(p) = \beta_0 \prod_{i=1}^{M} p_i \beta_i$$  (5.35)

Since $E(\lambda p, u) = \lambda E(p, u)$, the following relation apply,

$$\sum_{i=1}^{M} \alpha_i = 1$$

$$\sum_{i=1}^{M} r_{ij}^* = \sum_{j=1}^{M} r_{ij}^* = \sum_{i=1}^{M} \beta_i = 0$$  (5.36)

By Shephard’s Lemma.

$$\frac{\partial \log E}{\partial \log p_i} = \frac{\partial E}{\partial p_i} \cdot \frac{p_i}{E} = \frac{\partial E}{\partial p_i} \cdot \frac{x_i p_i}{E} \equiv s_i$$  (cost share $i$)  (5.37)

So that cost share equations are attained from Shephard’s Lemma by differentiating (5.34)–(5.35):
\[ s_i = \frac{\partial \log E}{\partial \log p_i} \]
\[ = \alpha_i + \sum_j r^*_ij \log p_j + \frac{u}{\log E - \nu(2) \text{ by (??)}} \cdot \frac{\partial b}{\partial \log p_i} \quad (5.38) \]

This (5.38) simplifies to
\[ s_i = \alpha_i + \sum_{i=1}^N r_{ij} \log p_i + \beta_i \log Y/P \quad (5.39) \]

where \( Y \) is consumer expenditure \((E)\) and \( P \) is a price index given by
\[ \log P = \alpha_0 + \sum_{j=1}^M \log p_i + \frac{1}{2} \sum_{i=1}^M \sum_{j=1}^M r_{ij} \log p_i \log p_j \quad (5.40) \]

and
\[ r_{ij} = \frac{1}{2} (r^*_ij + r^*_ji) \quad \text{for all } i, j \quad (5.41) \]

Except for the price index \( P \), demands (5.39) are linear is coefficients. Homogeneity and symmetric imply.
\[ \sum_{j=1}^M r_{ij} = 0 \quad i = 1, \cdots, M \quad \text{(homogeneity)} \quad (5.42a) \]
\[ r_{ij} = r_{ji} \quad \text{for all } i, j = 1, \cdots, M. \quad (5.42b) \]

In practice \( P \) is usually approximated by an appropriate arbitrary pure index, e.g.
\[ \log P \approx \sum_{j=1}^M s_j \log p_j \quad (5.43) \]

and then (5.39) is estimated.

### 5.5 Functional forms for short-run cost functions

\[ c(w, y, K) \quad \text{CRTS } f(\lambda x, \lambda K) = \lambda c(x, K) \Rightarrow c(w, \lambda y, \lambda K) = \lambda c(w, y, K) \]

The issue: form for \( c(w, y, K) \) should be 2nd-order flexible functional form with and without CRTS.
5.5.1 Normalized quadratic: $c^* = c/w_0$, $w^* = w/w_0$.

e.g.
\[
c^* = \left( a_0 + \sum_i a_i w_i^* + \frac{1}{2} \sum_i \sum_j a_{ij} w_i^* w_j^* \right) y + \left( b_0 + \sum_i b_i w_i^* \right) y^2
+ \left( c_0 + \sum_i c_i w_i^* \right) K + \left( d_0 + \sum_i d_i w_i^* \right) K^2 + \left( e_0 + \sum_i e_i w_i^* \right) \sqrt{Ky}
\]
\[\Rightarrow \text{(under CRTS)}\]
\[
c^* = \left( a_0 + \sum_i a_i w_i^* + \frac{1}{2} \sum_i \sum_j a_{ij} w_i^* w_j^* \right) y + \left( c_0 + \sum_i c_i w_i^* \right) K + \left( e_0 + \sum_i e_i w_i^* \right) \sqrt{Ky}
\]

5.5.2 Generalized Leontief:

\[
c = \left( \frac{1}{2} \sum_i \sum_j a_{ij} \sqrt{w_i} \sqrt{w_j} \right) y + \left( \sum_i b_i w_i \right) y^2 + \left( \sum_i c_i w_i \right) K + \left( \sum_i d_i w_i \right) K^2
+ \left( \sum_i e_i w_i \right) \sqrt{Ky}
\]
\[\Rightarrow \text{(under CRTS)}\]
\[
c = \left( \frac{1}{2} \sum_i \sum_j a_{ij} \sqrt{w_i} \sqrt{w_j} \right) y + \left( \sum_i c_i w_i \right) K + \left( \sum_i e_i w_i \right) \sqrt{Ky}
\]

5.5.3 Translog:

\[
\log c = a_0 + \sum_i a_i \log w_i + \frac{1}{2} \sum_i \sum_j a_{ij} (\log w_i)(\log w_j) + b_0 \log y + b_1 (\log y)^2
+ \sum_i b_i (\log w_i)(\log y) + c_0 \log K + c_1 (\log K)^2 + \sum_i c_i (\log w_i)(\log K) + e(\log y)(\log K)
\]
\[\Rightarrow \text{(under CRTS)}\]
\[
\log c(w, \lambda y, \lambda K) = \log \lambda + \log c(w, y, K)
\]

References

Chapter 6
Aggregation Across Agents in Static Models

6.1 General properties of market demand functions

In previous lecture we characterized the behavior of an individual producer or consumer at a static equilibrium. However in practice we often have data aggregated over agents rather than data for the individual producers or consumers. This leads to the following question: do the behavioral restrictions that apply to data at the firm or consumer level also apply to data that has been aggregated over agents?

In several simple behavioral models the answer to this question is “yes”. The most obvious example is the simple theory of static profit maximization, where all input are free variable and all firms face the same prices \( w, p \). Then the market prices \( w, p \) correspond exactly to the parameter facing the individual firms so that industry output supply and factor demand relations

\[
Y = Y(w, p),
X = X(w, p),
\]

where

\[
Y = \sum_{f=1}^{F} y^f, \quad X = \sum_{f} x^f,
\]
do not misrepresent the parameters facing individual firms. More precisely,

\[
\pi^f(w, p) - \left\{ py^f - \sum_{i=1}^{N} w_i x_i^f \right\} \geq 0 \text{ for all } (w, p) \quad f = 1, \cdots, F
\]

\[
\Rightarrow \sum_{f} \pi^f(w, p) - \sum_{f} \left\{ py^f - \sum_{i=1}^{N} w_i x_i^f \right\} \geq 0 \text{ for all } (w, p)
\]

i.e.

\[
\Pi(w, p) - \left\{ pY - \sum_{i=1}^{N} w_i X_i \right\} \geq 0 \text{ for all } (w, p)
\]

By (6.1), we can develop the properties of the industry profit function \( \Pi(w, p) \) and industry output supplies \( Y = Y(w, p) \) and derives demands \( X = X(w, p) \).
in essentially the same manners as in the case of data for the individual firm (see section 4.1 of lecture 4 on nonlinear duality).\footnote{The one exception concerns whether these relations \( \pi(w, p), y(w, p), x(w, p) \) are well defined in cases of free entry and exit to the industry (see footnote 2 on page 19).}

As a second example consider the case where each consumer maximizes utility subject to the value \( \sum_i p_i w_i^f \) of his initial endowments \( w_i^f = (w_1^f, \cdots, w_N^f) \) of the \( N \) commodities rather than an income \( y^f \) that is independent of commodity prices \( p \), i.e. each firm \( f \) solves

\[
\max_{x^f} u^f(x^f) \\
\text{s.t. } \sum_{i=1}^N p_i x_i^f = \sum_{i=1}^N p_i w_i^f.
\]

Then the solution \( x^* \) to (6.2) is conditional on \( (p, w; F) \) and in turn aggregate market demands \( \sum f x_f^* \) are conditional on \( (p, w^1, \cdots, w^F) \). If the endowments \( w_f \) of consumer \( f \) are constant over time for each consumer \( f = 1, \cdots, F \), then for purposes of estimation the market demands \( \sum f x_f^* \) are in effect conditional only on the prices \( p \) which are common to each consumer: \( x^* = x^*(p) = \sum f x_f^*(p) \). Then the aggregate market demand relation \( x = x(p) \) are well defined and inherit all linear restrictions on \( x_f(p) \). This includes the homogeneity restrictions \( x_f(\lambda p, y^f) = x_f(p, y^f) \), which can be expressed as \( x_f(\lambda p, w^f) = x_f(p, w^f) \), but apparently excludes the nonlinear second order conditions

\[
\left[ x_i^f p_j (p, y^f) - x_i^f y_j^f (p, y^f) x_j^f \right] = \left[ x_i^f p_j (p, u^f) \right] \text{ symmetric negative semidefinite (6.3)}
\]

which implies the Slutsky relations

\[
x_p^f (p, y^f) + x_y^f (p, y^f) x_f^f (p, y^f) \text{ symmetric negative semidefinite (6.3)}
\]

(property (3.4).b on page 31). It has been shown that, in the absence of special restrictions on utility functions \( u_f(x^f) \) or on the distribution of income \( y^f \) over consumers, the above restrictions (6.3) on demands of the individual consumer do not carry over to aggregate demands \( X = X(p, Y) \) where \( X = \sum f x_f^f \) and \( Y = \)
\[ \sum_f y^f. \] Indeed, if the number of consumers is equal to or greater than the number of goods \( N \), then any continuous function \( X(p, Y) \) that satisfies Walras’ Law

Walras Law is a principle in general equilibrium theory asserting that when considering any particular market, if all other markets in an economy are in equilibrium, then that specific market must also be in equilibrium. Walras Law hinges on the mathematical notion that excess market demands (or, conversely, excess market supplies) must sum to zero. That is, \( \sum XD = \sum XS = 0 \). Walras’ Law is named for the mathematically inclined economist Leon Walras, who taught at the University of Lausanne, although the concept was expressed earlier but in a less mathematically rigorous fashion by John Stuart Mill in his Essays on Some Unsettled Questions of Political Economy (1844). (in particular the adding up properties \( \sum_{i=1}^N \frac{\partial x_i(p, y)}{\partial y} p_i = 1, \sum_{i=1}^N \frac{\partial x_i(p, y)}{\partial p_j} p_i + \frac{\partial x_i(p, y)}{\partial y} y = 0, j = 1, \ldots, N \) in the differentiable case) can be generated by some set of utility maximizing consumers with some distribution of income (see Debreu 1974, pp. 15–22; Sonnenschein 1973, pp. 345–354).

There is a relatively simple way of addressing the above question. Following Diewert (1977, page 353–362) write the Slutsky relations (6.3) in the form

\[ x_p^f(p, y^f) = K_f^{N \times N} - x_y^f(p, y^f) x_f^f(p, y^f) \]

where \( K^f \equiv x_p^f(p, u^f) \) is the Hicksian substitution matrix which is symmetric negative semidefinite. In general \([x_f^{y_j}(p, y^f)x_f^f(p, y^f)]\) is neither symmetric nor negative semidefinite. Choose \( N - 1 \) linearly independent vector \( v_1, v_2, \ldots, v_{N-1} \) such that \( x_f^f v_n = 0 \) for \( n = 1, \ldots, N - 1 \), and write \( v_1, \ldots, v_{N-1} \) in matrix form as \( V \). Pre and post-multiplying (6.4) by \( V \),

\[ V^T x_p^f = V^T K_f^f V - V^T x_y^f x_f^f V = V^T K^f V \]

since \( x_f^f V = 0 \). Since \( K^f \equiv x_p^f(p, u^f) \) is symmetric negative semidefinite, (6.5) implies that \( V^T x_p^f(p, y^f)V^f \) is symmetric negative semidefinite.

Now sum (6.4) over consumers \( f = 1, \ldots, F \) to obtain the matrix of partial derivatives of aggregate demands \( X = \sum_f x^f \) with respect to prices \( p \) as

\[ X_p(p, y^f) = \sum_f K_f - \sum_f x_y^f(p, y^f) x_f^f(p, y^f). \]

Let \( \tilde{v}_1, \ldots, \tilde{v}_{N-F} \) be \( N - F \) linearly independent vectors, each of which is orthogonal to \( x^1, \ldots, x^F \), the set of initial demand vectors of the \( F \) consumers. Define the \( N \times (N - M) \) matrix \( \tilde{V} = [\tilde{v}_1, \ldots, \tilde{v}_{N-M}] \). Pre and post-multiplying (6.6) by \( \tilde{V} \),

\[ \tilde{V}^T X_p \tilde{V} = \tilde{V}^T K_f \tilde{V} - \tilde{V}^T x_y^f x_f^f \tilde{V} = \tilde{V}^T K_f \tilde{V} \]

Let \( \tilde{v}_1, \ldots, \tilde{v}_{N-F} \) be \( N - F \) linearly independent vectors, each of which is orthogonal to \( x^1, \ldots, x^F \), the set of initial demand vectors of the \( F \) consumers. Define the \( N \times (N - M) \) matrix \( \tilde{V} = [\tilde{v}_1, \ldots, \tilde{v}_{N-M}] \). Pre and post-multiplying (6.6) by \( \tilde{V} \),

\[ \tilde{V}^T X_p \tilde{V} = \tilde{V}^T K_f \tilde{V} - \tilde{V}^T x_y^f x_f^f \tilde{V} = \tilde{V}^T K_f \tilde{V} \]
since $\tilde{V}^T X f = 0_{N-F}$ for all $f = 1, \cdots, F$. Therefore $K^f$ symmetric negative semidefinite $(f = 1, \cdots, F)$ implies

$\tilde{V}^T X p \tilde{V}$ is a symmetric negative semidefinite matrix of dimension $(N-F) \times (N-F)$. \hfill (6.8)

(6.8) can be viewed as restrictions places on aggregate consumer demands $X(p; y^1, \cdots, y^F)$ by symmetry negative semidefiniteness of Hicksian demand responses $x^f_p (p, u^f)$ $(f = 1, \cdots, F)$. Moreover these are essentially the only restrictions on the first derivatives of market demand functions (aside from adding up constraints) (Mantel 1977). Note that if the number of consumers $F$ is equal to the number of goods $N$, then (6.8) places 0 restrictions on aggregate demands (the dimensions of the matrix of restrictions are $(N-F) \times (N-F)$ equals $0 \times 0$ in this case). This is consistent with the result of Debreu and Sonnenschein that symmetry negative semidefiniteness of Hicksian demand responses $x^f_p (p, u^f) \equiv K^f$ places no restrictions on aggregate demands when the number of consumers is equal to or greater than number of goods.

In sum, in general (i.e. in the absence of restrictions on production/utility functions or on the distribution of exogenous parameters across agents) the microtheory of the individual agent cannot be applied to data that has been aggregated over agents. In the next two sections we investigate how restrictions on production/utility functions and on the distribution of parameters across agents can influence this conclusion.

### 6.2 Condition for exact linear aggregation over agents

Here we consider restrictions on production and utility functions that imply consistent aggregation over agents for any distribution of the exogenous parameters that vary over agents. First, consider the conditional factor demands

$$x^f_i = x^f_i (w, y^f) \quad i = 1, \cdots, N \quad f = 1, \cdots, F \quad (6.9)$$

where $w \equiv$ vector of factor prices (which are common to each firm) and $y^f \equiv$ output level of firm $f$. Aggregate factor demands $\sum_f x^f = X(w, \sum_f y^f)$ exist if and only if

$$X_i \left( w, \sum_f y^f \right) = \sum_f x^f_i (w, y^f) \quad i = 1, \cdots, N \quad (6.10)$$

for all data $(w, y^1, \cdots, y^F)$. Differentiating (6.10) w.r.t. $y^f$, \hfill
6.2 Condition for exact linear aggregation over agents

\[ \frac{\partial X_i(w, y)}{\partial y} \frac{\partial y}{\partial y_f} = \frac{\partial x_i^f(w, y^f)}{\partial y_f} \]

i.e. \[ \frac{\partial X_i(w, y)}{\partial y} = \frac{\partial x_i^f(w, y^f)}{\partial y_f} \quad i = 1, \ldots, N \quad f = 1, \ldots, F \] (6.11)

since \( \partial y/\partial y_f = 1 \) in the case of linear aggregation \( y = \sum_f y^f \).

Thus, in the absence of restrictions on the distribution of total output \( y \) over firms, aggregate factor demands \( X(w, \sum_f y^f) \) exist if and only if \( \partial x_i^f(w, y^f)/\partial y_f \) are constant across all firms (for a given \( w \)). This condition implies that conditional factor demand equations for firms are of the following form:

\[ x_i^f = \alpha_i(w)y^f + \beta_i^f(w) \quad i = 1, \ldots, N \quad f = 1, \ldots, F \] (6.12)

where the function \( \alpha_i(w) \) is invariant over firms. (6.12) is called a “Gorman Polar Form”

Gorman polar form is a functional form for indirect utility functions in economics. Imposing this form on utility allows the researcher to treat a society of utility-maximizers as if it consisted of a single individual. W. M. Gorman showed that having the function take Gorman polar form is both a necessary and sufficient for this condition to hold. Here the scale effects \( \partial x_i^f(w, y^f)/\partial y_f \) are independent of the level of output, which implies that the production function \( y^f = y^f(x^f) \) is “quasi-homothetic”:

Fig. 6.1 quasi-homothetic
production function

Here, as output \( y^f \) expands and factor price \( w \) remain constant, the cost minimizing level of inputs increase along a ray (straight line) in input space. Condition (6.12) implies that aggregate demands have a Gorman Polar Form\(^2\):

\[^2\text{The more restrictive assumption of homotheticity implies that the expansion path } \frac{\Delta x_i^f(w, y^f)}{\Delta y_f} \text{ is a ray through the origin. Given the Gorman Polar Form (6.12), homotheticity requires } \beta^f(w) = 0 \text{ (which in turn implies the stronger assumption of constant restriction to scale).}\]
\[ X_i = \sum_f x^f_i (w, y^f) = \alpha_i (w) \sum_f y^f + \sum_f \beta^f_i (w) = \alpha_i (w) Y + \beta_i (w) \]

\[ i = 1, \cdots, N. \tag{6.13} \]

Condition (6.12) implies the existence of aggregate conditional factor demands satisfying (6.10), irrespective of whether producers are minimizing cost. The next question is: under what conditions do the restrictions on cost minimizing factor demands at the firm level generalize to aggregate factor demands \( X(w, \sum_f y^f) \)? The simplest way to answer this question is as follows: a Gorman Polar Form for aggregate demands (6.13) implies that identical demand relations could be generated by a single producer with an output level equal to \( \sum_f y^f \), and by assumption this producer is a cost minimizer. Therefore aggregate demands of a Gorman Polar Form necessarily inherit the properties of cost minimizing factor demand for a single agent.

To answer the above question more rigorously, first note that the assumption of cost minimization and condition (6.12) for existence of aggregate factor demands \( X(w, \sum_f y^f) \) are jointly equivalent to the assumption of cost minimization plus a Gorman Polar Form for the cost function of firm \( f \):

\[ c^f (w, y^f) = a(w) y^f + b^f (w), \quad f = 1, \cdots, F. \tag{6.14} \]

Sufficiency is obvious: (6.14) and Shephard’s Lemma imply \( x^f_i (w, y^f) = \frac{\partial c^f (w, y^f)}{\partial w_i} = \frac{\partial a(w)}{\partial w_i} y^f + \frac{\partial b^f (w)}{\partial w_i} \) where \( \frac{\partial a(w)}{\partial w_i} = \alpha_i (w) \), \( \frac{\partial b^f (w)}{\partial w_i} = \beta_i (w) \), and necessity follows simply by integrating (6.12) up to a cost function using Shephard’s Lemma. This implies the existence of an industry cost function which is also a Gorman Polar Form:

\[ \sum_f c^f (w, y^f) = \sum_f \left( a(w) y^f + b^f (w) \right) \\
= a(w) \sum_f y^f + \sum_f b^f (w) \\
= a(w) \sum_f y^f + b(w) \tag{6.15} \]

Moreover this industry cost function \( C(w, \sum_f y^f) = a(w) \sum_f y^f + b(w) \) inherits all the properties of cost minimization by a firm. To see this, note that
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\[ c^f \left( w, y^f (x^f) \right) - \sum_{i=1}^{N} w_i x_i^f \leq 0 \quad \text{for all} \quad w \quad f = 1, \cdots, F \]

\[ \Rightarrow \sum_{f} c^f \left( w, y^f (x^f) \right) - \sum_{f} \sum_{i=1}^{N} w_i x_i^f \leq 0 \quad \text{for all} \quad w \]

\[ \Leftrightarrow a(w) \sum_{f} y^f (x^f) + \sum_{f} b^f (w) - \sum_{f} \sum_{i=1}^{N} w_i x_i^f \leq 0 \quad \text{for all} \quad n \quad (6.16) \]

\[ \Leftrightarrow a(w) \sum_{f} y^f + b(w) - \sum_{i=1}^{N} w_i x_i \leq 0 \quad \text{for all} \quad w \]

\[ \Leftrightarrow C \left( w, \sum_{f} y^f \right) - \sum_{i=1}^{N} w_i x_i \leq 0 \quad \text{for all} \quad w \]

\[ \text{and likewise} \]

\[ c^f \left( w, y^f (x^f) \right) - \sum_{i=1}^{N} w_i x_i^f = 0 \quad f = 1, \cdots, F \]

\[ \Rightarrow C \left( w, \sum_{f} y^f \right) - \sum_{i=1}^{N} w_i x_i = 0 \quad (6.17) \]

Thus the aggregate primal-dual relation \( C(w, \sum_{f} y^f) - \sum_{i=1}^{N} w_i x_i \) has the same properties as the cost minimizing firm’s primal-dual \( c^f (w, y^f) - \sum_{i=1}^{N} w_i x_i^f \) \((f = 1, \cdots, F)\), and in turn \( C(w, \sum_{f} y^f) \) has the same properties as the cost minimizing \( c^f (w, y^f) \):

**Property 6.1.**

a) \( C(w, \sum_{f} y^f) \) is increasing in \( w, \sum_{f} y^f \);

b) \( C(\lambda w, \sum_{f} y^f) = \lambda C \left( w, \sum_{f} y^f \right) \);

c) \( C(w, \sum_{f} y^f) \) is concave in \( w \);

d) \( C(w, \sum_{f} y^f) \) satisfies Shephard’s Lemma

\[ X_i \left( w, \sum_{f} y^f \right) = \frac{\partial C \left( w, \sum_{f} y^f \right)}{\partial w_i} \quad i = 1, \cdots, N \]

---

\(^3\) Conditions Prop.6.1.a–c are satisfied for \( C(w, \sum_{f} y^f) = a(w) \sum_{f} y^f + b(w) \) if \( a(w) > 0 \) and both \( a(w) \) and \( b(w) \) are increasing, linear homogeneous and concave in \( w \).
In turn the aggregate demands satisfy the restrictions as the cost minimizing demands for a firm:

Property 6.2.

\[ a) \ X (\sum_f y^f) = X (\sum_f y^f) \]

\[ b) \ \frac{\partial X (\sum_f y^f)}{\partial w} \text{ is symmetric negative semidefinite.} \]

Similar results hold for linear aggregation of consumer demand equations \( x^f_i = x^f_i (p, y^f) \) where \( p \) price of consumer goods (which are assumed to be identical for all consumers) and \( y^f \equiv \text{exogenous income of consumer } f \). The conditions for existence of aggregate demands are

\[ X_i \left( p, \sum_f y^f \right) = \sum_f x^f_i \left( p, y^f \right) \quad i = 1, \cdots, N \] (6.18)

and the conditions are equivalent to Gorman Polar Forms

\[ x^f_i \left( p, y^f \right) = a_i (p) y^f + \beta^f_i (p) \quad i = 1, \cdots, N \quad f = 1, \cdots, F \] (6.19)

which imply

\[ X_i \left( p, \sum_f y^f \right) = a_i (p) \sum_f y^f + \beta_i (p) \quad i = 1, \cdots, N. \] (6.20)

In order to determine rigorously the restrictions that permit aggregate demands (6.20) to inherit the properties of utility maximization, first note that utility maximization by a consumer is essentially equivalent to cost minimization (see page 28):

\[ \min_{x^f} \sum_{i=1}^{N} p_i x^f_i = E^f (p, u^f) \] (6.21)

\[ \text{s.t.} \quad u^f (x^f) = u^{f*} \]

where \( u^{f*} \) is the maximum attainable utility level for the consumer given his budget constraint \( px^f = y^f \). An aggregate cost function \( E(p, \sum_f u^f) \) exists if and only if

\[ E \left( p, \sum_f u^f \right) = \sum_f E^f (p, u^f) \] (6.22)

and this condition is equivalent to the existence of Gorman Polar Forms

\[ E^f (p, u^f) = a(p) u^f + b^f (p) \quad f = 1, \cdots, F \] (6.23)
which in turn implies

\[ E\left(p, \sum_f u^f\right) = a(p) \sum_f u^f + b(p) \quad (6.24) \]

(6.23) and (6.24) imply

\[ E^f\left(p, u^f(x^f)\right) - \sum_{i=1}^{N} p_i x_i^f \leq 0 \quad \text{for all } p \quad f = 1, \cdots, F \]

\[ \Rightarrow E\left(p, \sum_f u^f\right) - \sum_{i=1}^{N} p_i x_i \leq 0 \quad \text{for all } p \quad (6.25) \]

\[ E^f\left(p, u^f(x^f)\right) - \sum_{i=1}^{N} p_i x_i^f = 0 \quad f = 1, \cdots, F \]

\[ \Rightarrow E\left(p, \sum_f u^f\right) - \sum_{i=1}^{N} p_i x_i = 0. \]

Therefore \( E(p, \sum_f u^f) \) inherit the properties of cost minimization, and Shephard’s Lemma implies that

\[ X^h_i\left(p, \sum_f u^f\right) = \frac{\partial E\left(p, \sum_f u^f\right)}{\partial p_i} = \frac{\partial a(p)}{\partial p_i} \sum_f u^f + \frac{\partial b(p)}{\partial p_i} \quad i = 1, \cdots, N \quad (6.26) \]

In order to relate (6.26) to Marshallian demands, note that (6.23) and utility maximization by consumers \( f \) imply

\[ y^f = a(p)u^f* + b^f(p) \]

\[ \Rightarrow u^f* = \frac{y^f - b^f(p)}{a(p)} \quad f = 1, \cdots, F \quad (6.27) \]

i.e. the consumer’s inherit utility function has the Gorman Polar Form

\[ V(p, y^f) = \tilde{a}(p)y^f + \tilde{b}^f(p) \quad (6.28) \]

where \( \tilde{a}(p) = 1/a(p), \tilde{b}(p) = -b^f(p)/a(p). \) Substituting (6.27) into (6.26),
\[ X_i = \frac{\partial a(p)}{\partial p_i} \sum_f \frac{y^f - b^f(p)}{a(p)} \]
\[ = \frac{\partial a(p)}{\partial p_i} \frac{1}{a(p)} \sum_f y^f - \frac{\partial a(p)}{\partial p_i} \sum_f \frac{b^f(p)}{a(p)} + \frac{\partial b(p)}{\partial p_i} \quad i = 1, \cdots, N \]
\[ (6.29) \]

implies the existence of aggregate Marshallian demands of a Gorman Polar Form:
\[ X_i \left(p, \sum_f y^f\right) = \tilde{a}(p) \sum_f y^f + \tilde{b}(p) \quad i = 1, \cdots, N \]
\[ (6.30) \]

where \( a(p) = \frac{\partial a(p)}{\partial p_i} \), \( b(p) = -\frac{\partial a(p)}{\partial p_i} + \frac{\partial b(p)}{\partial p_i} \).

Thus cost minimization by individual consumers and a Gorman Polar Form (6.18/6.30) that satisfy restrictions analogous to restriction on Marshallian demands by individual utility maximizing consumers. The restriction (6.1) implied by cost minimization are satisfies for \( E(p, \sum_f u^f) \) (6.24) if
\[ a(p) > 0; \]
\[ a(p), b(p) \text{ increasing and linear homogeneous in } P; \]
\[ \left[ \frac{\partial^2 a(p)}{\partial p \partial p}, \left[ \frac{\partial^2 b(p)}{\partial p \partial p} \right] \right] \text{ symmetric, negative semidefinite.} \]
\[ (6.31) \]

These restrictions can be tested or imposed on the Gorman Polar Form \( X(p, \sum_f y^f) \) (6.30)

Alternatively we could try to incorporate the utility maximization restrictions into aggregate demands \( X(p, \sum_f y^f) \) directly by specifying consumers’ indirect utility functions \( V^f(p, y^f) \) as Gorman Polar Form:
\[ V^f(p, y^f) = \tilde{a}(p)y^f + \tilde{b}(p). \]
\[ (6.32) \]

However inverting this relation yields
\[ y^f = u^f \frac{-\tilde{b}(p)}{-\tilde{a}(p)} \]
\[ (6.33) \]

i.e.
\[ E^f(p, u^f) = \frac{1}{\tilde{a}(p)} u^f - \tilde{b}(p) \quad f = 1, \cdots, F. \]
\[ (6.34) \]

Thus an indirect utility function \( V^f(p, y^f) \) has a Gorman Polar Form if and only if the corresponding cost function \( E^f(p, u^f) \) has a Gorman Polar Form. Therefore the above analysis in term of cost minimizing behavior exhaust the restrictions per-
mitting the existence of aggregate Marshallian demands \( X(p, \sum_f y^f) \) that satisfy the same restrictions as Marshallian demands \( x^f(p, y^f) \) for an individual utility maximizing consumer.

Note that these restrictions (6.23–6.24, 6.30–6.31) on preference structure do imply mild restriction on the distribution of expenditures over consumers:

\[
E^f = a(p)u^f + b^f(p) \quad a(p) > 0 \\
\implies y^f > b^f(p) \quad f = 1, \ldots, F
\]  

(6.35)

Aside from this restriction, Gorman Polar Forms permit aggregate demands to inherit the properties of utility maximizing demands irrespective of the distribution of expenditure over consumers. \( b^f(p) \) can be viewed as the agent’s “committed expenditure” at prices \( p \) (since it is independent of utility level \( u^f \)), and \( y^f - b^f(p) \) can be defined as the corresponding “uncommitted expenditure”.

Aggregation problems also arise when there are variations in prices over agents, and these problems can be severe. For example, suppose that profit maximizing competitive firms in an industry are distributed across different regions of the country and as a result firms face different output prices \( p^f \). Aggregate demands can be defined as \( \sum_f x^f = X(w, \sum_f \gamma_f p^f) \), \( \sum_f y^f = Y(w, \sum_f \gamma_f p^f) \) where \( \sum_f \gamma_f p^f \) denotes a (weighted) average output price \( \bar{p} \). Aggregate demands and supplies exist if

\[
X_i \left( w, \sum_f \gamma_f p^f \right) = \sum_f x_i^f \left( w, p^f \right) \quad i = 1, \ldots, N \\
Y \left( w, \sum_f \gamma_f p^f \right) = \sum_f y^f \left( w, p^f \right)
\]  

(6.36)

These conditions are satisfied if the aggregate demands and supplies have Gorman Polar Forms:

\[
X_i = a_i(w) \sum_f \gamma_f p^f + \beta_i(w) \quad i = 1, \ldots, N \\
Y = a_0(w) \sum_f \gamma_f p^f + \beta_0(w)
\]  

(6.37)

Now suppose further that these aggregate relation are to inherit the properties of profit maximizing demand and supply relations for the individual firm. In the absence of any restrictions on the distribution of prices \( p^f \) over firms, this requires the firm and industry profit functions to have the following Gorman Polar Forms

\[
\pi^f \left( w, p^f \right) = a(w)p^f + \beta^f(w) \quad f = 1, \ldots, F \\
\Pi \left( w, \sum_f \gamma_f p^f \right) = a(w) \sum_f \gamma_f p^f + b(w)
\]  

(6.38)

However Hotelling’s Lemma now implies that output supplies are independent of output prices:
In sum, Gorman Polar Forms (GPF) are both necessary and sufficient for exact linear aggregation over agents. Unfortunately these functional forms are fairly restrictive. For example, GPF conditional factor demands \( x = x(w, y) \) imply linear expansion paths, and GPF consumer demands \( x = x(p, y) \) imply linear Engel curves (and income elasticities tend to unity as total expenditure increases). These assumptions may or may not be realistic over large changes in output or expenditure (see Deaton and Muellbauer 1980, pp. 144–145, 151–153).

On the other hand it should be noted that Gorman Polar Form cost and indirect utility functions are flexible functional forms. These former provide second order approximations to arbitrary cost and indirect utility functions (Diewert 1980, pp. 595–601). Therefore Gorman Polar Forms provide a local first order approximation to any system of demand equations (except, of course, for cases such as (6.38)).

### 6.3 Linear aggregation over agents using restrictions on the distribution of output or expenditure

The analysis in the previous section was aimed at achieving consistent aggregation over agents while imposing essentially zero restrictions on the distribution of output or expenditure over agents. However these distribution are in fact highly restricted in most cases, and these restrictions may help to achieve consistent aggregation.

Unfortunately there are few results on the relation between restrictions on the distribution of “exogenous” variables such as output or income and consistent aggregation. This section simply summarizes several example that illustrate the effects of alternative restrictions on the distribution of such variables.

First, output or expenditure may be choice variables for the agent rather than exogenous variables, and it is very important to incorporate this fact into the analysis of possibilities for consistent aggregation. For example, suppose that producers are competitive profit maximizers and face the same output price \( p \). Then the first order condition in the output market for profit maximization is

\[
\frac{\partial c^f(w, y^f)}{\partial y^f} = p \quad f = 1, \ldots, F
\]

(6.40)

where \( c^f(w, y^f) = \min_{x} \sum_{i=1}^{N} w_i x_i \) s.t. \( y^f(x^f) = y^f \). This implies that the marginal cost is identical across firms at all observed combinations of output levels \( (y_1^f, \ldots, y_F^f) \) for given prices \( w, p \).

Now remember from our earlier discussion that identical marginal cost across firms is the condition for consistent linear aggregation of cost functions across firms.
6.3 Linear aggregation over agents using restrictions on the distribution of output or expenditure (irrespective of the distribution of output across firms) (see pages 64–66). Thus, the assumption of competitive profit maximization and identical price imply that an aggregate cost function $C(w, \sum_f y_f^*)$ exists over all profit maximizing levels of output $\sum_f y_f^* = \sum_f y_f(w, p)$. Moreover this aggregate cost function $C(w, \sum_f y_f)$, where $\sum_f y_f$ is restricted to the equilibrium levels $\sum_f y_f(w, p)$, also inherits the properties of cost minimization.

As a second example, supposed that cost functions of individual consumers are of the form

$$E_f(p, u_f) = a_f(p)u_f + b_f(p) \quad f = 1, \ldots, F. \quad (6.41)$$

this is slightly more general than the Gorman Polr Form (6.23) in the sense that have the function $a_f(p)$ can vary over consumers. Nevertheless preference are still quasi-homothetic.

Solving $y_f = a_f(p)u_f^* + b_f(p)$ (6.41) for the consumer’s indirect utility function yields

$$V_f(p, y_f) = \frac{y_f - b_f(p)}{a_f(p)} \quad f = 1, \ldots, F, \quad (6.42)$$

and applying Roy’s Theorem to this result (6.42) yields

$$x_{f}^i(p, y_f) = \frac{-\frac{\partial b_f(p)}{\partial p_i} - \frac{\partial a_f(p)}{\partial p_i} (y_f - b_f(p))}{a_f(p)^2} \quad (6.43)$$

Thus, in contract to the Gorman Polar Form,

$$\frac{\partial x_{f}^i(p, y_f)}{\partial y_f} = \frac{\partial a_f(p)}{\partial p_i} \quad i = 1, \ldots, N \quad f = 1, \ldots, F \quad (6.44)$$

i.e. Engel curves can vary over consumers (although these curves are still linear).

Now suppose that the distribution of “uncommitted expenditure” remains proportionally constant over consumers, i.e. consumer incomes $y_1, \ldots, y_F$ satisfy the following restrictions for all variations in commodity prices $p$:

---

4 On the other hand, endogenizing consumer expedition $y_f$, as the wage rate $w$ times the amount of labor supplied by the agent, is not sufficient for consistent linear aggregation in the case of utility maximization. Endogenizing $y_f$ in this manner implies the additional first order condition $\partial u_f(x_f^*, x_{Le}^*)/\partial x_{Le} = \left[\partial V_f(p, y_f^*)/\partial y_f\right]w$ where $x_{Le} = le$ of leisure for consumer $f$, $(f = 1, \ldots, F)$. Since the marginal utility of leisure $\partial u_f(x_f^*, x_{Le}^*)/\partial x_{Le}$ will generally vary over consumers, the marginal utility of income $\partial V_f(p, y_f^*)/\partial y_f$ also varies over consumers.
\[ y^f - b^f(p) = \lambda^f \left[ \sum_{f=1}^{F} \left( y^f - b^f(p) \right) \right] > 0 \quad f = 1, \cdots, F \]

where \( \lambda^f > 0 \), \( \sum_{f=1}^{F} \lambda^f = 1 \).

If the individual cost functions are of the form (6.41) and if the distribution of expenditure over agents satisfies the restriction (6.45), then aggregate Marshallian demand functions exist, have Gorman Polar Form and inherit the properties of utility maximization.

**Proof.** Summing (6.43) over consumers and substituting in (6.45),

\[ \sum_{f} x_i^f = \sum_{f} \frac{\partial b^f(p)}{\partial p_i} + \sum_{f} \frac{\partial a^f(p)}{\partial p_i} \lambda^f \left[ \sum_{f} \left( y^f - b^f(p) \right) \right] / a^f(p) \]

\[ = \sum_{f} \frac{\partial b^f(p)}{\partial p_i} + \sum_{f} \lambda^f \frac{\partial a^f(p)}{\partial p_i} / a^f(p) \sum_{f} y^f - \sum_{f} \lambda^f \frac{\partial a^f(p) / \partial p_i}{a^f(p)} \sum_{f} b^f(p) \]

\[ = \frac{\partial \beta(p)}{\partial p_i} + \left( \frac{\partial \alpha(p) / \partial p_i}{\alpha(p)} \right) \left( \sum_{f} y^f - \beta(p) \right) \quad i = 1, \cdots, N \]

(6.46)

where \( \beta(p) \equiv \sum_{f} b^f(p) \), \( \alpha(p) \equiv \prod_{f=1}^{F} a^f(p)^{\lambda^f} \). Thus there exist aggregate Marshallian demand functions of Gorman Polar Form. A GPF implies that aggregate demands are identical to those that could be generated by a single consumer with expenditure \( \sum_{f} y^f \), and by assumption this consumer maximizes utility. Thus the aggregate demand relations (6.46) inherit the properties of utility maximization. \( \square \)

We have the following corollary to the above result.

**Corollary 6.1.** Suppose that the utility function of individual consumers are homothetic, so that the expenditure function of individual consumers have the form

\[ E^f(p, u^f) = a^f(p) u^f \quad f = 1, \cdots, F \]

(6.47)

and also suppose that each consumer’s share \( \lambda^f \) in total expenditure \( \sum_{f} y^f \) is fixed over the data set. Then aggregate Marshallian demand relations \( \sum_{f} x_i^f = X(p, \sum_{f} y^f) \) exist, have the form

\[ \sum_{f} x_i^f = \left( \frac{\partial \alpha(p) / \partial p_i}{\alpha(p)} \right) \sum_{f} y^f \quad i = 1, \cdots, N \]

(6.48)

(using the notation of (6.46)) and inherit the properties of utility maximization. In order to see that this is a special case of the result proves above, simply note that (6.47) is a special case of (6.41) where \( b^f(p) \equiv 0 \), and that \( b^f(p) \equiv 0 \) \( (f = 1, \cdots, F) \) reduce (6.45) to
6.4 Condition for exact nonlinear aggregation over agents

\[ y^f = \lambda^f \sum_{f=1}^{F} y^f > 0 \quad f = 1, \ldots, F \]  
\[ \text{where } \lambda^f > 0, \sum_{f=1}^{F} \lambda^f = 1 \]  
\[ (6.49) \]

In one respect the assumption of homotheticity in (6.47) is more restrictive than the quasi-homothetic Gorman Polar form (6.23) at constant price \( p \) the ratio of cost minimizing demands is independent of the level of expenditure, i.e. increases in expenditure \( y^f \) lead to increases in consumption of all commodities along a ray from the origin rather than from an arbitrary point. On the other hand, (6.47) is more general than Gorman Polar Form (6.23) in the sense that the function \( a^f(p) \) can vary over agent, so that (linear) Engel curves can vary over agents.

6.4 Condition for exact nonlinear aggregation over agents

The previous sections assumed that aggregate output or expenditure \( Y \) is to be constructed as a simple linear sum \( Y = \sum_{f} y^f \) of the outputs of expenditure \( y^f \) of individual agents. More generally, the aggregation relation can be written as

\[ Y = Y(y^1, \ldots, y^F). \]  
\[ (6.50) \]

Then the conditions for existence of an aggregate cost function for producers (for example) can be written as

\[ C \left( w, Y(y^1, \ldots, y^F) \right) = \sum_{f} c^f(w, y^f) \]  
\[ (6.51) \]

for all \( (w, y^1, \ldots, y^F) \). Differentiating (6.51) w.r.t. \( y^f \),

\[ \frac{\partial C(w, Y)}{\partial Y} \frac{\partial Y}{\partial y^f} = \frac{\partial c^f(w, y^f)}{\partial y^f} \quad f = 1, \ldots, F \]  
\[ (6.52) \]

which implies

\[ \frac{\partial Y}{\partial y^f} / \frac{\partial y}{\partial y^g} = \frac{\partial c^f(w, y^f)}{\partial y^f} / \frac{\partial c^g(w, y^g)}{\partial y^g} \quad f, g = 1, \ldots, F. \]  
\[ (6.53) \]

Condition (6.53) implies that \( Y = Y(y^1, \ldots, y^F) \) is strongly separable (see Deaton and Muellbauer 1980, pp. 137–142), so that

\[ Y = Y \left( \sum_{f} h^f(y^f) \right). \]  
\[ (6.54) \]
The corresponding aggregate cost function and aggregate demands are

\[
C(w, Y(y^1, \cdots, y^F)) = \alpha(w) Y \left( \sum_f h^f(y^f) \right) + \beta(w)
\]

\[
X_i(w, Y(y^1, \cdots, y^F)) = \frac{\partial \alpha(w)}{\partial w_i} Y \left( \sum_f h^f(y^f) \right) + \frac{\partial \beta(w)}{\partial w_i} \quad i = 1, \cdots, N.
\]

(6.55)

The above cost function is more general than the Gorman Polar Form since the aggregate output \(Y\) is not restricted to the linear case \(\sum_f y^f\). The aggregate cost function inherits the properties of cost minimization, so the aggregate factor demands \(X(w, Y(y^1, \cdots, y^F))\) can be derived from the aggregate cost function using Shephard’s Lemma. These conditions for exact nonlinear aggregation are less restrictive than Gorman Polar Form, but it is not easy to make this distinction operational (see Deaton and Muellbauer 1980, pp. 154–158, for one attempt).

References

Chapter 7

Aggregation Across Commodities: Non-index Number Approaches

Consumers and also producers generally use a wide variety of commodities, so substantial aggregation (grouping) of commodities is necessary to make econometrics studies manageable. This is particularly the case with flexible functional forms, where the number of parameters to be estimated increases exponentially with the number of commodities that are modeled explicitly. Presumably aggregation over commodities generally misrepresent the choices faced by consumers and producers, so the microeconomic theory that applies to the true behavioral model with disaggregated commodities may not generalize to a model with highly aggregated commodities.

This leads to the following questions: when does aggregation over commodities not misrepresent the agent’s choices or behavior, and how is this aggregation procedure defined? This lecture summaries two types of results related to this question: the composite commodity theorem and conditions for two stage budgeting. The composite commodity theorem of Hicks demonstrates that certain restrictions on the covariation of prices permit consistent aggregation, and the discussion of two stage budgeting shows that separability restrictions (plus other restrictions) on the structure of utility functions, production functions etc. also permit consistent aggregation over commodities.¹

The next lecture provides a more satisfactory answer to the above question. Certain index number formulas for aggregation over commodities can be rationalized in terms of certain functional forms for production functions, cost functions, etc., including popular second order flexible functional forms. Thus certain index number formulas for aggregation over commodities inherit the desirable properties of approximation that characterize the corresponding second order flexible functional forms.

¹ For simplicity we will assume a single agent, i.e. we will abstract from problems in aggregating over agents.
7.1 Composite commodity theorem

If the prices of several commodities are in fixed propositions over a data set, then these commodities can be correctly treated as a single composite commodity with one price. For example, suppose that a consumer maximizes utility over three commodities \( x_1, x_2, x_3 \) and that prices \( p_2 \) and \( p_3 \) remain in fixed proportion over the data set, i.e.

\[
p_{2,t} = \theta_t p_{2,0} \quad p_{3,t} = \theta_t p_{3,0} \quad \text{for all times } t \tag{7.1}
\]

where \( p_{2,0} \) and \( p_{3,0} \) are base period prices of commodities 2 and 3, and \( \theta_t \) is a (positive) scalar that varies over time \( t \). Equivalently the consumer can be viewed as solving a cost minimization problem

\[
E (p_1, p_2, p_3, u) = \min \sum_{i=1}^{3} p_i x_i \quad \text{s.t. } u(x) = u^* \tag{7.2}
\]

Combining (7.1) and (7.2),

\[
E (p_{1,t}, p_{2,t}, p_{3,t}, u_t) = E (p_{1,t}, \theta_t p_{2,0}, \theta_t p_{3,0}, u_t)
\]

\[
= \tilde{E} (p_{1,t}, \theta_t u_t) \quad \text{for all } t \tag{7.3}
\]

Cost minimizing behavior (7.2) implies that \( \tilde{E} (p_1, \theta, u) \) inherits all the properties of a cost function, including Shephard’s Lemma:

\[
\frac{\partial \tilde{E}(p_1, \theta, u)}{\partial p_1} = x_1 \\
\frac{\partial \tilde{E}(p_1, \theta, u)}{\partial \theta} = \frac{\partial}{\partial \theta} (p_1 x_1 + \theta p_{2,0} x_2 + \theta p_{3,0} x_3) \\
= p_{2,0} x_2 + p_{3,0} x_3 \tag{7.4}
\]

by the envelope theorem.

Thus \( p_{2,0} x_{2,t} + p_{3,0} x_{3,t} \) can be interpreted as the quantity \( x^c \) of a composite commodity at time \( t \), and \( \theta_t \) is the price of the corresponding composite commodity at time \( t \). The resulting system of Hicksian demands

\[
x_1 = x^h_1 (p_1, \theta, u) \\
x^c = p_{2,0} x_2 + p_{3,0} x_3 = x^h (p_1, \theta, u) \tag{7.5}
\]

inherit the properties of cost minimizing demands. Therefore the corresponding system of demands

\[
x_1 = x_1 (p_1, \theta, y) \\
x^c = p_{2,0} x_2 + p_{3,0} x_3 = x^c (p_1, \theta, y) \tag{7.6}
\]
inherit the properties of utility maximizing demands.

However, this composite commodity theorem may be of limited value in justifying and defining aggregation over commodities. Even if two commodities are perceived as relatively close substitutes in consumption, their relative prices may vary significantly due to differences in the supply schedules of the two commodities.

7.2 Homothetic weak separability and two-stage budgeting

The assumption of “two-stage budgeting” has often been used to simplify studies of consumer behavior. The general utility maximization problem (3.1) can be written as

\[
\begin{align*}
\max_{x} & \quad u(x_1, \ldots, x_A, x_B, \ldots, x_{N_A}, \ldots, x_1, \ldots, x_Z, \ldots, x_{N_B}) = V(p, y) \\
\text{s.t.} & \quad \sum_{i=1}^{N_A} p_i^A x_i^A + \sum_{i=1}^{N_B} p_i^B x_i^B + \cdots + \sum_{i=1}^{N_Z} p_i^Z x_i^Z = y 
\end{align*}
\]

where \(x = (x_1, \ldots, x_A, x_B, \ldots, x_{N_A}, \ldots, x_1, \ldots, x_Z, \ldots, x_{N_B})\), i.e. there are \(N_A + N_B + \cdots + N_Z\) commodities. Two-stage budgeting can then be outlined as follows.

In the first stage total expenditures \(y\) are allocated among broad groups of commodities \(x\). For example the consumer decides to allocate the expenditures \(y^A\) to commodities \(x^A = (x_1^A, \ldots, x_N^A)\), \(y^B\) to commodities \(x^B = (x_1^B, \ldots, x_{N_B}^B)\), \ldots, \(y^Z\) to commodities \(x^Z = (x_1^Z, \ldots, x_{N_Z}^Z)\), where \(y^A + y^B + \cdots + y^Z = y\). This allocation of expenditures among broad groups requires knowledge of total expenditures \(y\) and of an aggregate price \(p^A, p^B, \ldots, p^Z\) for each group of commodities.

Thus in the first stage a consumer is viewed as solving a problem of the form

\[
\begin{align*}
\max_{\tilde{x}} & \quad u(\tilde{x}^A, \tilde{x}^B, \ldots, \tilde{x}^Z) \\
\text{s.t.} & \quad \tilde{p}^A \tilde{x}^A + \tilde{p}^B \tilde{x}^B + \cdots + \tilde{p}^Z \tilde{x}^Z = y 
\end{align*}
\]

where \(\tilde{x} = (\tilde{x}^A, \ldots, \tilde{x}^Z)\) and \(\tilde{p} = (\tilde{p}^A, \ldots, \tilde{p}^Z)\) denote aggregate commodities and prices for the broad groups \(A, \ldots, Z\). Alternatively, utilizing the equivalence between utility maximization and cost minimization (see pages 27 to 28), in the first stage a consumer can be viewed as solving a cost minimization problem of the form

\[
\begin{align*}
\min_{\tilde{x}} & \quad \tilde{p}^A \tilde{x}^A + \tilde{p}^B \tilde{x}^B + \cdots + \tilde{p}^Z \tilde{x}^Z \\
\text{s.t.} & \quad u(\tilde{x}) = u^* 
\end{align*}
\]

In the second stage the group expenditure \(y^A, y^B, \ldots, y^Z\) are allocated among the commodities within each group. Here the consumer solves maximization problems of the form
max \( u^A(x^A_1, \ldots, x^A_{N_A}) \) \( \ldots \) \( \max u^Z(x^Z_1, \ldots, x^Z_{N_Z}) \)

\[ \text{s.t. } \sum_{i=1}^{N_A} p^A_i x^A_i = y^A \] \[ \ldots \] \[ \text{s.t. } \sum_{i=1}^{N_Z} p^Z_i x^Z_i = y^Z \] \( (7.10) \)

where \( u^A(x^A_1, \ldots, x^A_{N_A}), \ldots, u^Z(x^Z_1, \ldots, x^Z_{N_Z}) \) are interpreted as “sub-utility function” for the commodities within each group \( A; \ldots, Z \).

What restrictions on the consumer’s utility function \( u(x) \) imply that two-stage budgeting is realistic, i.e. under what restrictions do the general utility maximization problem \( (7.7) \) and the two-stage procedure \( (7.8) \) and \( (7.10) \) yield the same solutions \( x^* \)? First consider the second stage of two-stage budgeting. Define a weakly separable utility function as follows:

**Definition 7.1.** A utility function \( u(x) \) is defined as “weakly separable” in commodity groups \( x^A = (x^A_1, \ldots, x^A_{N_A}), \ldots, x^Z = (x^Z_1, \ldots, x^Z_{N_Z}) \) if and only if \( u(x) \) can be written as

\[ u(x) = \tilde{u} \left[ u^A(x^A_1, \ldots, x^A_{N_A}), \ldots, u^Z(x^Z_1, \ldots, x^Z_{N_Z}) \right] \]

over all \( x \) for some macro-utility function \( u = \tilde{u}(u^A, \ldots, u^Z) \) and sub-utility function \( u^A = u^A(x^A_1, \ldots, x^A_{N_A}), \ldots, u^Z = u^Z(x^Z_1, \ldots, x^Z_{N_Z}) \).

The second stage of two-stage budgeting is equivalent to the restriction that the consumer’s utility function \( u(x) \) is weakly separable. To be more precise,

**Property 7.1.** The general utility maximization problem \( (7.7) \) and a series of “second stage” maximization problems \( (7.10) \) (conditional on group expenditure \( y^A, \ldots, y^Z \)) yield the same solution \( x^* \) if and only if \( u(x) \) is weakly separable in the above manner in Definition 7.1.

**Proof.**

First, suppose that \( u(x) \) is weakly separable as in Definition 7.1 and that \( \partial u / \partial u^A > 0, \ldots, \partial u / \partial u^Z > 0 \). Then utility maximization \( (7.7) \) requires that each subutility \( u^A, \ldots, u^Z \) be maximized conditional on its group expenditure \( y^A, \ldots, y^Z \). For example if \( x^{A*} \) solving Definition 7.1 does not solve \( u^A(x^A) \) s.t. \( \sum_{i=1}^{N_A} p^A_i x^A_i = y^A \), then \( y^A \) could be reallocated among commodities \( x^A \) so as to increase \( u^A \) without decreasing \( u^B, \ldots, u^Z \), i.e. so as to increase the total utility level \( u \) without violating the budget constraint \( px = y \). Thus the second stage of two-stage budgeting is satisfied if \( u(x) \) is weakly separable. Second, suppose that two-stage budgeting is satisfied. Two-stage budgeting implies \( x^{A*} = x^A(p, y) \) solving \( (7.7) \) can be written as

\[ x^{A*}_i = x^A_i(p^A, y^A) \quad i = 1, \ldots, N_A \] \( (7.11) \)

and similarly for subgroups \( B, \ldots, Z \). Without loss of generality \( x^{A*} \) solves
7.2 Homothetic weak separability and two-stage budgeting

\[
\max_{x^A} u \left(x^A, x^{B*}, \ldots, x^{Z*}\right) \\
\text{s.t. } \sum_{i=1}^{N_A} p_i^A x_i^A = y^A 
\]  

(7.12)

where \(x^B, \ldots, x^Z\) are fixed at their equilibrium levels solving (7.7), (7.11) implies that (given \(p^A, y^A\)) \(x^{A*}\) is independent of \(p^B, \ldots, p^Z\) and hence independent of \(x^{B*}, \ldots, x^{Z*}\). Thus (7.12) reduces to (7.10), i.e. there exists a subutility function \(u^A(x^A)\) for commodity group \(A\) that is independent of the levels of other commodities \(x^B, \ldots, x^Z\). Thus two-stage budgeting implies weak separability \(u \left(u^A(x^A), \ldots, u^Z(x^Z)\right)\). □

Second, consider the first stage of two-stage budgeting. The critical point here is to be able to construct a price \(\tilde{p}^A, \ldots, \tilde{p}^Z\) for each commodity group \(A, \ldots, Z\) such that a first stage utility maximization or cost minimization problem yields the optimal allocation of expenditure across subgroups \(A, \ldots, Z\).

Given weak separability of \(u(x)\), a sufficient condition for the first stage is homotheticity of each subutility function \(u^A(x^A), \ldots, u^Z(x^Z)\). To be more precise,

\[\text{Property 7.2. If } u(x) \text{ is weakly separable and each subutility function } u^A(x^A), \ldots, u^Z(x^Z) \text{ is homothetic, then there exists a first stage maximization problem that obtains the same distribution of group expenditures } y^A, \ldots, y^Z \text{ as in the general case (7.7).} \]

\[\text{Proof.} \] Weak separability of \(u(x)\) implies second stage maximization (7.10) and equivalently a series of cost minimization problems.

\[
\min_{x^A} \sum_{i=1}^{N_A} p_i^A x_i^A = E^A(p^A, u^{A*}) \\
\text{s.t. } u^A(x^A) = u^{A*} \\
\vdots \\
\min_{x^Z} \sum_{i=1}^{N_Z} p_i^Z x_i^Z = E^Z(p^Z, u^{Z*}) \\
\text{s.t. } u^Z(x^Z) = u^{Z*} 
\]

(7.13)

where \(u^{A*} = u^A(x^{A*}), \ldots, u^{Z*} = u^Z(x^{Z*})\) for \(x^{A*}, \ldots, x^{Z*}\) solving (7.7). Under weak separability the consumer’s general cost minimization problem \(E(p, u^*) = \min_x p x \text{ s.t. } u(x) = u^*\) can be restated as
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\[
E(p, u^*) = \min_{u^A, \ldots, u^Z} A \left( p^A, u^A \right) + \cdots + E \left( p^Z, u^Z \right)
\]
\[
\text{s.t. } \tilde{u} \left( u^A, \ldots, u^Z \right) = u^*
\]
(7.14)

\(u^A(x^A)\) homothetic implies \(E^A(p^A, u^A) = \phi(u^A)E^A(p^A)\) (see Property 1.3.a on page 6) where without loss of generality we can define \(\phi(u^A) = u^A\) (constant returns to scale), since the indexing of the indifference curves representing the consumer’s preferences is arbitrary. Therefore under homotheticity (7.14) reduces to

\[
E \left( p, u^* \right) = \min_{u^A, \ldots, u^Z} A e^A \left( p^A \right) + \cdots + u^Z e^Z \left( p^Z \right)
\]
\[
\text{s.t. } \tilde{u} \left( u^A, \ldots, u^Z \right) = u^*
\]
(7.15)

This can be interpreted as a first stage cost minimization problem with aggregate prices \(c^A(p^A), \ldots, c^Z(p^Z)\) and leading to the following optimal allocation of expenditures \(y\) over subgroups:

\(y^A = u^* e^A(p^A), \ldots, y^Z = u^* e^Z(p^Z)\)

Weak separability places substantial restrictions on the degree of substitution between commodities in different groups. Since weak separability implies that utility maximizing demands for a group of commodities \(x^A\) depends only on prices \(p^A\) and group expenditure \(y^A\) (7.11), it follows that other prices \(p^B\) influence demands \(x^A\) only through changes in the optimal level of expenditure \(y^A\) for the group. This implies the following restriction on the Hicksian substitution effects across groups:

\[
\frac{\partial x^A_i(p, u^*)}{\partial p^B_j} = \varphi^{AB}(p, y) \frac{\partial x^A_i(p^A, y^A)}{\partial y^A} \frac{\partial x^B_j(p^B, y^B)}{\partial y^B}
\]
\[
i = 1, \ldots, N_A \quad j = 1, \ldots, N_B
\]
(7.16)

where the function \(\varphi^{AB}(p, y)\) is independent of the choice of commodities \(i, j\) from group \(A, B\) (see Deaton and Muellbauer, pp. 128–129 for a sketch of a proof), moreover, this relationship (7.16) is both necessary and sufficient for weak separability of groups \(A\) and \(B\), i.e. restrictions (7.16) exhaust the implications of weak separability.

The most obvious application of this discussion of two-stage budgeting is in the estimation of, e.g., consumer demand for food. In general the demand for food goods \(x^A = (x_1^A, \ldots, x_{N_A}^A)\) will depend on the prices of all food and non-food final goods and on total expenditure \(y\). On the other hand suppose that the consumer’s utility function \(u(x)\) is weakly separable in food commodities \(x^A\) and all other commodities \(x^B\), i.e. \(u(x) = \tilde{u}(u^A(x^A), u^B(x^B))\). Then the demand equations for food \(x^A = x^A_i(p^A, p^B, y)\) \((i = 1, \ldots, N_A)\) can be simplified to \(x^A_i = x^A_i(p^A, y^A)\) \((i = 1, \ldots, N_A)\) (7.11) which can be derived from a subutility maximization problem.
7.3 Implicit separability and two-stage budgeting

$$\max_{x^A} u^A(x^A) = V^A(p^A, y^A)$$

s.t. $$\sum_{i=1}^{N_A} p_i^A x_i^A = y^A$$  \hspace{1cm} (7.17)

Here $V^A(p^A, y^A)$ has the properties of an indirect utility function and the corresponding Marshallian demands $x^A = x^A(p^A, y^A)$ inherit the properties of utility maximization.

One complication in estimating the above model (7.17) for food demand is that food expenditure $y^A$ is not exogenous to the consumer, and ignoring this fact generally leads to biases in estimation. In the absence of any further assumptions (beyond weak separability) $y^A$ generally depends on all prices and total expenditure, i.e. $y^A = y^A(p^A, p^B, y)$.

However if the subutility functions $u^A(x^A)$ and $u^B(x^B)$ are homothetic (implying two-stage budgeting), then the corresponding expenditure functions can be written as $A^A e^A(p^A)$ and $A^B e^B(p^B)$ and the relation $y^A = y^A(p^A, p^B, y)$ can be rewritten as

$$y^A = y^A \left[ e^A(p^A), e^B(p^B), y \right]$$  \hspace{1cm} (7.18)

Thus we can posit homothetic functional forms $E^A = u^A e^A(p^A)$ and $E^B = u^B e^B(p^B)$, derive the corresponding indirect utility function $V^A = y^A / e^A(p^A)$ and differentiate this indirect utility function to obtain the estimating equations $x^A = x^A(p^A, y^A)$, and use these functional forms $e^A(p^A), e^B(p^B)$ as the aggregate prices in (7.18). Unfortunately the expenditure equation (7.18) still depends on prices $p^B$ as well as $p^A$, but at least the estimating equation (7.18) is more restricted than the general equation $y^A = y^A(p^A, p^B, y)$.

### 7.3 Implicit separability and two-stage budgeting

An alternative two-stage budgeting procedure can be obtained if certain separability restrictions hold for the consumer’s cost function $E(p, u)$ rather than for his utility function $u(x)$. Preferences are defined as “implicitly separable” in broad groups $A, \cdots, Z$ if the cost function can be written in the form

$$E(p, u) = \tilde{E} \left[ e^A(p^A, u), \cdots, e^Z(p^Z, u), u \right]$$  \hspace{1cm} (7.19)

where $p^A = (p_1^A, \cdots, p_{N_A}^A), \cdots, p^Z = (p_1^Z, \cdots, p_{N_Z}^Z)$. Note that total utility $u$ (rather than subutilities $u^A, \cdots, u^Z$) appear in each of the cost function $e^A(p^A, u), \cdots, e^Z(p^Z, u)$, so there are no group subutilities in contrast to the case where $u(x)$ is weakly separable.

---

\(^2\) Alternatively we can eliminate $y^A$ from the demand equations $x^A = x^A(p^A, y^A)$ using the identity $y^A = \sum_{i=1}^{N_A} p_i^A x_i^A$. 

---
The following two-stage budgeting procedure is defined by simple differentiation of the macrofunction \( \tilde{E}(e^A, \cdots, e^Z) \) and then the group cost function \( e^A(p^A, u), \cdots, e^Z(p^Z, u) \):

\[
\text{(first stage) } \quad s^A = \frac{\sum_{i=1}^{N_A} p^A_i x^A_i}{y^A} = \frac{\partial \log \tilde{E}(e^A, \cdots, e^Z, u)}{\partial \log e^A} \\
\vdots \\
\text{(second stage) } \quad s^Z = \frac{\sum_{i=1}^{N_Z} p^Z_i x^Z_i}{y^Z} = \frac{\partial \log \tilde{E}(e^A, \cdots, e^Z, u)}{\partial \log e^Z} \\
\]

(7.20a)

(7.20b)

where \( y^A = y^A + \cdots + y^Z \) (i.e. total expenditure) and \( y^A = \sum_{i=1}^{N_A} p^A_i x^A_i \) (expenditure of group \( A \)), etc. Thus the budget shares \( s^A, \cdots, s^Z \) of groups \( A, \cdots, Z \) are obtained by simple logarithmic differentiation of the macrofunction \( \tilde{E}(e^A, \cdots, e^Z, u) \), and the share \( s^A(i = 1, \cdots, N_A), \cdots, s^Z(i = 1, \cdots, N_Z) \) of each commodity in group expenditure is obtained by simple logarithmic differentiation of the group cost functions \( e^A(p^A, u), \cdots, e^Z(p^Z, u) \).

**Proof.** Differentiating (7.19) w.r.t. \( p^A_i \) and applying Shephard’s Lemma,

\[
\frac{\partial E(p, u)}{\partial p^A_i} = \frac{\partial \tilde{E}(\cdot) \partial e^A(\cdot)}{\partial e^A(p^A)} = x^A_i(p, u) \quad i = 1, \cdots, N_A.
\]

(7.21)

Thus

\[
y^A = \sum_{i=1}^{N_A} x^A_i p^A_i = \frac{\partial \tilde{E}(\cdot)}{\partial e^A} \sum_{i=1}^{N_A} \frac{\partial e^A(\cdot)}{\partial p^A_i} p^A_i = \frac{\partial \tilde{E}(\cdot)}{\partial e^A} e^A(\cdot) \quad \text{by Euler’s Theorem}
\]

(7.22)
since $e^A(p^A, u)$ is linear homogeneous in $p^A$.

$$s^A = \frac{y^A}{y}$$

$$= \frac{\partial \tilde{E}(\cdot) / \partial e^A}{y / e^A(\cdot)}$$

$$= \frac{\partial \log \tilde{E}(\cdot)}{\partial \log e^A}$$

(7.23)

which is (7.20a). Substituting $\frac{\partial \tilde{E}(\cdot)}{\partial e^A} = \frac{y^A}{e^A(\cdot)}$ (7.22) into $\frac{\partial e^A(\cdot)}{\partial p^i} \frac{\partial \tilde{E}(\cdot)}{\partial e^A} = x_i^A$ (7.21) yields (7.20b).

Note that implicit separability is sufficient for the two-stage budgeting procedure outlined in (7.20). In contrast weak separability of $u(x)$ was sufficient only for the second stage of the budgeting procedure discussed in the previous section. Also note that implicit separability implies that the ratio of commodities within a group is independent of prices of commodities outside the group, i.e.

$$\frac{\partial \left[x_i^A(p, u)/x_j^A(p, u)\right]}{\partial p_k^B} = 0 \quad i, j = 1, \cdots, N_A \quad k = 1, \cdots, N_B$$

(7.24)

(see (7.21)). In contrast, weak separability of $u(x)$ implied that the ratio of marginal rates of substitution between commodities within a group is independent of levels of commodities outside the group.

References


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\(^3\) $E(p, u)$ is linear homogeneous in $p$ only if $e^A(p^A, u), \cdots, e^Z(p^Z, u)$ are also linear homogeneous in prices and $\tilde{E}(e^A, \cdots, e^Z, u)$ is linear homogeneous in $e^A, \cdots, e^Z$. However note that $e^A(p^A, u)$ does not equal total expenditure $y^A$ on group $A$ (see (7.22)). Thus $c^A(p^A, u), \cdots, c^Z(p^Z, u)$ are to be interpreted as group price indexes that depend on utility level $u$. 
Chapter 8
Index Numbers and Flexible Functional Forms

In this lecture we show that particular index number formulas for aggregating over commodities can be rationalized in terms of particular functional forms for production functions or dual cost or profit functions. Approximately correct procedures for aggregating over commodities are presented for cases of Translog and Generalized Leontief functional forms. Since these functional forms provide a second order approximation to any true form, the corresponding index number formulas can often be interpreted as approximately correct.

The results obtained here should be contrasted with the previous lecture. There consistent aggregation over commodities was rationalized essentially in terms of assumptions of separability between groups of commodities. Here specific aggregation procedures are rationalized essentially in terms of specific functional forms for production functions or dual cost functions. Since assumptions of Translog or Generalized Leontief functional forms are usually considered less restrictive than assumptions of (homothetic) weak separability, this lecture presents a more useful basis for a theory of approximate aggregation over commodities.

8.1 Laspeyres index numbers and linear functional forms

Until recently most index number computations have used simple base period weighting schemes, and the most common of these are Laspeyres quantity and price indexes. The Laspeyres quantity index can be written as

\[
\frac{X_1}{X_0} = \frac{\sum_{i=1}^{N} p_{i,0} x_{i,1}}{\sum_{i=1}^{N} p_{i,0} x_{i,0}}
\]  

(8.1)

where \( p_0 = (p_{1,0}, \ldots, p_{N,0}) \) denotes the prices for the \( N \) commodities in the base period \( t = 0 \), \( x_0 = (x_{1,0}, \ldots, x_{N,0}) \) denotes the quantities of the \( N \) commodities in the base period \( t = 0 \), and \( x_1 = (x_{1,1}, \ldots, x_{N,1}) \) denotes the quantities of the \( N \) commodities in any other period \( t = 1 \).
This aggregation procedure (8.1) can be defined as correct if the ratio of aggregates \( X_1, X_0 \) provides an accurate measure of the contributions of inputs \( 1, \ldots, N \) to the producer's output in different time periods. Thus in the case where aggregation is defined over all inputs (\( 1, \ldots, N \) is to be interpreted as all inputs) and there is a single output, the quantity index \( X_1/X_0 \) in (8.1) is correct if \( X_1/X_0 \) is equal to the ratio of outputs \( f(x_{1,1}, \ldots, x_{N,1})/f(x_{1,0}, \ldots, x_{N,0}) \) for any time periods \( t = 0, 1 \).

Similarly a Laspeyres price index can be written as

\[
P_1 \over P_0 = \frac{\sum_{i=1}^{N} x_{i,0} p_{i,1}}{\sum_{i=1}^{N} x_{i,0} p_{i,0}}
\]

where the prices \( p = (p_1, \ldots, p_N) \) for any period are weighted by the base period quantities \( x_0 = (x_{1,0}, \ldots, x_{N,0}) \). Interpreting commodities \( 1, \ldots, N \) as inputs in production and assuming cost minimizing behavior, this aggregation procedure can be defined as correct if the ratio of the aggregates \( P_1, P_0 \) provides an accurate measure of the contribution of inputs \( 1, \ldots, N \) to the cost of attaining a given level of output \( y \). In the case where aggregation is defined over all inputs (\( 1, \ldots, N \) is to be interpreted as all inputs), the price index \( P_1/P_0 \) is correct if \( P_1/P_0 \) is equal to the ratio of minimum costs \( C(p_{1,1}, \ldots, p_{N,1}, y)/C(p_{1,0}, \ldots, p_{N,0}, y) \) for any two time periods and a common output level \( y \).

Are the above aggregation procedures (8.1)–(8.2) correct for some cases of production functions? The answer is yes: assuming static competitive profit maximizing or cost minimizing behavior for a firm, an aggregation procedure (8.1) or (8.2) over inputs \( x = (x_1, \ldots, x_N) \) can be rationalized in terms of a linear production function with a family of parallel straight line isoquants

\[
x_2
\]

(8.3)

and also in terms of a linear production function with fixed coefficients, i.e. right angle isoquants
The first case (8.3) assumes perfect substitution between inputs and the second case (8.4) assumes zero substitution between inputs.

More formally, given Laspeyres quantity and price indexes $X_1 / X_0$ (8.1), a linear production function $y = f(x)$ satisfying either (8.3) or (8.4), and static competitive profit maximizing or cost minimizing behavior, then

$$\frac{X_1}{X_0} = \frac{f(x_{1,1}, \ldots, x_{N,1})}{f(x_{1,0}, \ldots, x_{N,0})} \quad \text{and} \quad \frac{P_1}{P_0} = \frac{c(p_{1,1}, \ldots, p_{N,1})}{c(p_{1,0}, \ldots, p_{N,0})} \quad \text{for all } t = 1, 0 \tag{8.5}$$

where $c(p)$ denotes a unit cost function (the minimum cost of producing one unit of output $y$ given prices $p$) (see Diewert 1976 pp. 182–183 for a proof of (8.5)). Under the above assumptions $X_1 / X_0$ is the ratio of output and $P_1 / P_0$ is the ratio of unit cost of output for the two periods $t = 1, 0$. In this sense Laspeyres quantity and price indexes are exact for both a linear production function satisfying (8.3) and a linear production function satisfying (8.4).

In either case linear production functions (with either zero or perfect substitution between inputs) are very unrealistic and highly restrictive. Linear production functions can provide only a first order approximation to an arbitrary production function. This suggests that Laspeyres indexes may often lead to substantial errors in aggregation over commodities, and it is unlikely that aggregate data collected in this manner inherits properties of profits maximization or cost minimization from disaggregate data. In the next two sections we obtain more positive results by demonstrating that particular quantity and price indexes are correct for Translog and Generalized Leontief functional forms.

### 8.2 Exact indexes for Translog functional forms

Economists have frequently advocated the use of Divisia indexes rather than Laspeyres indexes for aggregating over commodities (e.g. Hulten 1973, pp. 1017–1026). A

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1 It can easily be shown that Laspeyres indexes also provide a first order approximation to an arbitrary (true) index (see Deaton 1980, pp.173–174).
Divisia quantity index for (e.g.) inputs can be defined in continues time by the line integral
\[ \frac{X_t}{X_0} = \exp\left\{ \int \left( \sum_{i=1}^N s_i(t) \frac{\partial x_i(t)}{\partial t} \right) \right\} \]
where \( s_i(t) = \frac{w_i(t)x_i(t)}{\sum_{j=1}^N w_j(t)x_j(t)} \).

The following Törnqvist index is the most commonly used discrete approximation to the Divisia quantity index for inputs:
\[
\log \left( \frac{X_1}{X_0} \right) = \sum_{i=1}^N s_i \log \left( \frac{x_i,1}{x_i,0} \right) \tag{8.6}
\]
where
\[
s_i = \frac{1}{2} \left( \frac{w_{i,1}x_{i,1}}{\sum_{j=1}^N w_{j,1}x_{j,1}} + \frac{w_{i,0}x_{i,0}}{\sum_{j=1}^N w_{j,0}x_{j,0}} \right) \quad i = 1, \ldots, N.
\]

The Törnqvist quantity index (8.6) is exact for a Translog production function with constant returns to scale. In other words, assuming static competitive cost minimizing behavior,
\[
\log \left( \frac{X_t}{X_s} \right) = \log \left( \frac{\mathcal{f}(x_t)}{\mathcal{f}(x_s)} \right) \tag{8.7}
\]
for all periods \( s, t \) when \( \mathcal{f}(x) \) is Translog constant returns to scale and the quantity index is calculated as in (8.6). Such an index, which is exact for a constant returns (or variable returns) to scale flexible form for \( \mathcal{f}(x) \), is termed superlative.

**Proof.** First we establish an algebraic result for a quadratic function
\[
g(z) = a_0 + \sum_i a_i z_i + \frac{1}{2} \sum_i \sum_j a_{ij} z_i z_j \quad (a_{ij} = a_{ji} \text{ for all } i, j)
\]
or in matrix notation,
\[
g(z) = a_0 + a^T z + \frac{1}{2} z^T z A z \quad (A \text{ symmetric}) \tag{8.8}
\]
Then
\[
g(z_1) - g(z_0) = a^T (z_1 - z_0) + \frac{1}{2} z_1^T A z_1 - \frac{1}{2} z_0^T A z_0
\]
\[
= a^T (z_1 - z_0) + \frac{1}{2} z_1^T A (z_1 - z_0) + \frac{1}{2} z_0^T A (z_1 - z_0) \tag{8.9}
\]
since \( A \) is symmetric
\[
= \frac{1}{2} (a + Az_1 + a + Az_0)^T (z_1 - z_0)
\]
which implies
\[
g(z_1) - g(z_0) = \frac{1}{2} \left( \frac{\partial g(z_1)}{\partial z} + \frac{\partial g(z_0)}{\partial z} \right)^T (z_1 - z_0) \tag{8.10}
\]
Moreover (8.10) implies (8.8), i.e. (8.8) is correct if and only if (8.10) is correct. Since a Translog production function \( \log f = a_0 + \sum_{i=1}^{N} a_i \log x_i + \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} \log x_i \log x_j \) is quadratic in logs, (8.10) and \( \frac{\partial \log f(x)}{\partial \log x_i} = \frac{\partial f(x)}{f(x)} / x_i \) (\( i = 1, \ldots, N \)) imply

\[
\log f(x_1) - \log f(x_0) = \frac{1}{2} \sum_{i=1}^{N} \left( \frac{\partial f(x_1)}{\partial x_{i,1}} \frac{x_{i,1}}{f(x_1)} + \frac{\partial f(x_0)}{\partial x_{i,0}} \frac{x_{i,0}}{f(x_0)} \right) \log \left( \frac{x_{i,1}}{x_{i,0}} \right). \tag{8.11}
\]

Static competitive cost minimization implies \( \frac{\partial f(x)}{\partial x_i} = \frac{w_i}{\pi C(w,y)/y} \) (\( i = 1, \ldots, N \)), and constant returns to scale implies \( f(x) = \sum_{j=1}^{N} \frac{\partial f(x)}{\partial x_j} \cdot x_j \) (Euler’s theorem). Substituting these to (8.11),

\[
\log \left( \frac{f(x_1)}{f(x_0)} \right) = \frac{1}{2} \sum_{i=1}^{N} \left( \frac{w_{i,1} x_{i,1}}{\sum_{j=1}^{N} w_{j,1} x_{j,1}} + \frac{w_{i,0} x_{i,0}}{\sum_{j=1}^{N} w_{j,0} x_{j,0}} \right) \log \left( \frac{x_{i,1}}{x_{i,0}} \right) \tag{8.12}
\]

where right hand side is the Törnqvist input quantity index (8.6).

Moreover, the Törnqvist quantity index (8.6) is exact only for a Translog constant returns to scale production function (this follows from the equivalence between (8.10) and a quadratic function \( g(z) \) (8.8)).

The assumption of a constant returns to scale production function appears to be crucial to the interpretation of the particular Törnqvist quantity index (8.6) as exact. This assumption is not crucial only in the case of pair-wise comparisons of aggregate inputs, i.e. if the aggregate index (8.6) is to be calculated only for two time periods \( t = 0, 1 \) (see Dievert 1976, Op. cit.).

Nevertheless, a quantity index closely related to (8.6) can be interpreted as exact for a general Translog production function. Define the following quantity index:

\[
\log \left( \frac{X_1}{X_0} \right) = \frac{1}{2} \sum_{i=1}^{N} \left( \frac{w_{i,1} x_{i,1}}{p_{y,1} f(x_1)} + \frac{w_{i,0} x_{i,0}}{p_{y,0} f(x_0)} \right) \log \left( \frac{x_{i,1}}{x_{i,0}} \right) \tag{8.13}
\]

where \( p_{y,t} \equiv \text{price of the (single) output in period } t \). This deviates from the Törnqvist quantity index (8.6) only in that \( w_{i,t} x_{i,t} \) is divided by total revenue \( p_{y,t} f(x_t) \) rather than by total cost \( \sum_{j=1}^{N} w_{j,t} x_{j,t} \). In the case of constant returns to scale \( f(x) = \sum_{j=1}^{N} \frac{\partial f(x)}{\partial x_j} \cdot x_j \) (Euler’s theorem) and in turn \( p_y f(x) = \sum_{j=1}^{N} w_j x_j \). Thus (8.11) reduces to the Törnqvist index (8.6) in the case of constant returns to scale and competitive profit maximization. Unlike the index (8.6), the above quantity index (8.13) is exact for a general (variable returns to scale) Translog production function. This result requires the assumption of profit maximization rather than simple cost minimization, in contrast to (8.6).

**Proof.** Substituting the first order conditions for static competitive profit maximization \( \frac{\partial f(x_i)}{\partial x_i} = \frac{w_{i,t}}{p_{y,t}} \) (\( i = 1, \ldots, N \)) into (8.11),
\[
\log \left( \frac{f(x_1)}{f(x_0)} \right) = \frac{1}{2} \sum_{i=1}^{N} \left( \frac{u_{i,1}x_{i,1}}{P_{y,1}f(x_1)} + \frac{u_{i,0}x_{i,0}}{P_{y,0}f(x_0)} \right) \log \left( \frac{x_{i,1}}{x_{i,0}} \right) \tag{8.14}
\]

i.e. (8.13) is exact for the general Translog \( f(x) \).

This index (8.13) can be interpreted as an alternative Törnqvist index approximating the continuous Divisia quantity index.

Given an input quantity index \( X_1/X_0 \) such as (8.6) or (8.13), we can easily construct a corresponding implicit input price index using the following factor reversal equation:

\[
\left( \frac{W_1}{W_0} \right) \left( \frac{X_1}{X_0} \right) = \frac{\sum_{i=1}^{N} w_{i,1}x_{i,1}}{\sum_{i=1}^{N} w_{i,0}x_{i,0}} \tag{8.15}
\]

i.e. the product of the input price and quantity indexes is equal to the ratio of total expenditure on the \( N \) disaggregate inputs for the corresponding time periods \( t = 0, 1 \). Assuming \( X_1/X_0 = f(x_1)/f(x_0) \) (8.7), (8.15) implies

\[
\left( \frac{W_1}{W_0} \right) = \frac{\sum_{i=1}^{N} w_{i,1}x_{i,1}/f(x_1)}{\sum_{i=1}^{N} w_{i,0}x_{i,0}/f(x_0)} \tag{8.16}
\]

i.e. \( \left( W_1/W_0 \right) \) can be interpreted as the index of average costs of production. Obviously if a quantity index is superlative then the corresponding implicit price index is superlative in an analogous manner.

Alternatively an input price index can be calculated directly rather than by using (8.15). Now assume that the production function is constant returns to scale and that the cost function \( C(w, y) = y c(w) \) is Translog: \( \log c(w) = a_0 + \sum_{i=1}^{N} a_i \log w_i + \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} \log w_i \log w_j \). Define the following Törnqvist price index for inputs:

\[
\log \left( \frac{W_1}{W_0} \right) = \sum_{i=1}^{N} s_i \log \left( \frac{w_{i,1}}{w_{i,0}} \right) \tag{8.17}
\]

where \( s = (s_1, \ldots, s_N) \) are calculated as in (8.6).\(^2\) Proceeding as in the proof of (8.7), we obtain the result

\[
\log \left( \frac{W_1}{W_0} \right) = \log \left( \frac{c(w_{1,1}, \ldots, w_{N,1})}{c(w_{1,0}, \ldots, w_{N,0})} \right) \tag{8.18}
\]

which is analogous to (8.16). Thus the Törnqvist price index (8.17) is exact for a Translog unit cost function. In the absence of constant returns to scale in production, the assumption of a Translog cost function \( C(w, y) \) implies

\(^2\) Since flexible functional forms are not self-dual (e.g. a Translog production function does not imply a Translog cost function, or vice-versa), the price index (8.17) for a Translog unit cost function is not equivalent to the implicit price index (8.16) for a Translog constant returns to scale production function.
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\[ \log \left( \frac{W_1}{W_0} \right) = \sum_{i=1}^{N} s_i \log \left( \frac{w_i,1}{w_i,0} \right) \]
\[ = \log \left( \frac{C(w_{1,1}, \ldots, w_{N,1}, y_1)}{C(w_{1,0}, \ldots, w_{N,0}, y_0)} \right) \] (8.19)

i.e. the Törnqvist index (8.17) is equal to the ratio of total costs in different time periods \( t = 0, 1 \). Thus, in the absence of constant returns to scale, the direct Törnqvist index (8.17) cannot strictly be interpreted as a price index for inputs since it depends upon a measure \( y_1, y_0 \) of input quantities \( x_1, x_0 \) as well as upon input prices \( w_1, w_0 \).

In contrast, consider the index \( \frac{W_1}{W_0} \) calculated implicitly from (8.15) using the quantity index (8.13), science this index satisfies (8.16) for a general Translog production function and profit maximizing behavior, it can be interpreted as the ratio of average costs of production

\[ \log \left( \frac{W_1}{W_0} \right) = \log \left( \frac{C(w_{1,1}, \ldots, w_{N,1}, y_1)/y_1}{C(w_{1,0}, \ldots, w_{N,0}, y_0)/y_0} \right) \] (8.20)

even in the absence of constant returns to scale.

The derivations of exact index number equations are complicated somewhat in the case of multiple outputs \( y = (y_1, \ldots, y_M) \). Generalizations of the Translog production function do not appear to provide an entirely satisfactory basis for index number formulas. For example, assume a Translog transformation function

\[ \log y_1 = a_0 + \sum_{i=1}^{T} a_i \log z_i + \sum_{i=1}^{T} \sum_{j=1}^{T} a_{ij} \log z_i \log z_j \] (8.21)

where \( z = (y_2, \ldots, y_M, -x_1, \ldots, -x_N) \) \((T = M - 1 + N)\) and static competitive profit maximization

\[ \max_{y,x} \sum_{i=1}^{M} p_i y_i - \sum_{i=1}^{N} u_i x_i \]
\[ \text{s.t.} \quad y_1 = y_1(z) \] (8.22)

Then the quadratic identity (8.11) implies

\[ \log \left( \frac{y_{i,1}}{y_{i,0}} \right) = \sum_{i=1}^{N} s_i^x \log \left( \frac{x_{i,1}}{x_{i,0}} \right) - \sum_{i=2}^{M} s_i^y \log \left( \frac{y_{i,1}}{y_{i,0}} \right) \]

where \( s_i^x = \frac{1}{2} \left( \frac{w_{i,1}x_{i,1}}{p_{i,1}y_{i,1}} + \frac{w_{i,0}x_{i,0}}{p_{i,0}y_{i,0}} \right) \)
\[ s_i^y = \frac{1}{2} \left( \frac{p_{i,1}y_{i,1}}{p_{i,1}y_{i,1}} + \frac{p_{i,0}y_{i,0}}{p_{i,0}y_{i,0}} \right) \] (8.23)
(8.23) defines quantity indexes that are exact for the Translog transformation function (8.21). However the output quantity index \( \sum_{i=2}^{M} s_i^y \log \left( \frac{y_{i,1}}{y_{i,0}} \right) \) incorporates the level of the first output \( y_0 \) only indirectly via the definitions of the weights \( s_i^y \) for output \( 2, \ldots, M \).

A more satisfactory approach to the derivation of index numbers in the case of multiple outputs may be in terms of the cost function \( C(w, y) \). Assume a Translog joint cost function

\[
\log C = a_0 + \sum_{i=1}^{N} a_i \log w_i + \sum_{i=1}^{M} b_i \log y_i \\
+ \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} \log w_i \log w_j + \sum_{i=1}^{M} \sum_{j=1}^{M} b_{ij} \log y_i \log y_j \\
+ \sum_{i=1}^{T} \sum_{j=1}^{T} c_{ij} \log w_i \log y_j
\]

and static competitive profit maximizing behavior. Then the quadratic identity (8.11), Shephard’s Lemma and \( \frac{\partial C}{\partial y_i} = p_i \) \( (i = 1, \ldots, M) \) imply

\[
\log \left( \frac{C_1}{C_0} \right) = \sum_{i=1}^{N} s_i^x \log \left( \frac{w_{i,1}}{w_{i,0}} \right) + \sum_{i=1}^{M} s_i^y \log \left( \frac{y_{i,1}}{y_{i,0}} \right)
\]

where

\[
s_i^x = \frac{1}{2} \left( \frac{\sum_{i=1}^{N} w_{i,1} x_{i,1} + \sum_{i=1}^{N} w_{i,0} x_{i,0}}{\sum_{i=1}^{N} x_{i,1}} \right)
\]

\[
s_i^y = \frac{1}{2} \left( \frac{\sum_{i=1}^{M} p_{i,1} y_{i,1} + \sum_{i=1}^{M} p_{i,0} y_{i,0}}{\sum_{i=1}^{M} x_{i,1}} \right)
\]

The first sum \( \sum_{i=1}^{N} s_i^x \log \left( \frac{w_{i,1}}{w_{i,0}} \right) \) can be interpreted as a Törnqvist price index \( \log \left( \frac{W_1}{W_0} \right) \) for all inputs, and the second sum \( \sum_{i=1}^{M} s_i^y \log \left( \frac{y_{i,1}}{y_{i,0}} \right) \) can be interpreted as a Törnqvist quantity index \( \log \left( \frac{Y_1}{Y_0} \right) \) for all outputs. In order to see this, rewrite equation (8.25) as

\[
\frac{C_1}{C_0} = \left( \frac{W_1}{W_0} \right) \cdot \left( \frac{Y_1}{Y_0} \right)
\]

where

\[
\log \left( \frac{W_1}{W_0} \right) = \sum_{i=1}^{N} s_i^x \log \left( \frac{w_{i,1}}{w_{i,0}} \right)
\]

\[
\log \left( \frac{Y_1}{Y_0} \right) = \sum_{i=1}^{M} s_i^y \log \left( \frac{y_{i,1}}{y_{i,0}} \right)
\]

i.e. the index of total costs is always equal to the product of the input price index and the output quantity index. This confirms that \( \frac{Y_1}{Y_0} \) can be interpreted as an output
8.2 Exact indexes for Translog functional forms

The quantity index and \( W_1 / W_0 \) can be interpreted as an index of the contributions of inputs to the average cost of aggregate output.

The corresponding implicit Törnqvist input quantity and output price indexes can then be calculated from the equations

\[
\frac{W_1}{W_0} \left( \frac{X_1}{X_0} \right) = \frac{\sum_{i=1}^{N} w_{i,1} x_{i,1}}{\sum_{i=1}^{N} w_{i,0} x_{i,0}}
\]

\[
\frac{P_1}{P_0} \left( \frac{Y_1}{Y_0} \right) = \frac{\sum_{j=1}^{M} p_{j,1} y_{j,1}}{\sum_{j=1}^{M} p_{j,0} y_{j,0}}
\]

These implicit indexes \((X_1/X_0)\) and \((P_1/P_0)\) are also superlative. Finally, consider the problem of deriving index numbers for aggregation of commodities in the case of consumer behavior. Assuming that the utility function \( u(x) \) is homothetic (or equivalently constant returns to scale) and that the unit cost function \( e(p) = \min_{x} p x \) s.t. \( u(x) = 1 \) is Translog, then the Törnqvist price index

\[
\log \left( \frac{P_1}{P_0} \right) = \frac{1}{2} \sum_{i=1}^{N} \left( \frac{p_{i,1} x_{i,1}}{\sum_{j=1}^{N} p_{j,1} x_{j,1}} + \frac{p_{i,0} x_{i,0}}{\sum_{j=1}^{N} p_{j,0} x_{j,0}} \right) \log \left( \frac{p_{i,1}}{p_{i,0}} \right)
\]

and the corresponding implicit quantity index

\[
\log \left( \frac{X_1}{X_0} \right) = \log \left( \frac{\sum_{i=1}^{N} p_{i,1} x_{i,1}}{\sum_{j=1}^{N} p_{j,0} x_{j,0}} \right) - \log \left( \frac{P_1}{P_0} \right)
\]

are exact (see the discussion of (8.17)–(8.18)).

Alternatively suppose that the consumer’s utility function \( u(x) \) is Translog as well as homothetic:

\[
\log u = a_0 + \sum_{i=1}^{N} a_i \log x_i + \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} \log x_i \log x_j
\]

\( u(x) \) constant returns to scale implies that the indirect utility function \( V(p, y) \) is constant returns to scale in expenditure \( y \), so that (using Euler’s theorem) \( V(p, y) = \frac{\partial V(p, y)}{\partial y} \cdot y \). Applying the quadratic identity (8.10) to (8.30), and using the first order conditions \( \frac{\partial V(p, y)}{\partial y} \cdot p_i (i = 1, \ldots, N) \) and in turn \( \partial V(p, y)/\partial y = u/y \), we obtain the result

---

3 In the case of constant returns to scale in production, \( W_1 / W_0 \) can also be interpreted as a price index of the contributions of inputs to the marginal cost of aggregate output.
The right hand side of (8.31) defines a Törnqvist quantity index \( \log \left( \frac{X_1}{X_0} \right) \) which is exact for a homothetic Translog utility function \( u(x) \), and the corresponding implicit price index can be calculated using (8.29).

### 8.3 Exact indexes for Generalized Leontief functional forms

Results in the previous section demonstrated that many index numbers closely related to popular Törnqvist indexes can be rationalized in terms of underlying Translog functional forms. This section briefly illustrates that different index number formulas are implied by Generalized Leontief functional forms. Nevertheless, since both classes of functional forms provide a second order approximation to a true form, the various index number formulas should lead to similar results at least for small changes in quantities and prices.

First, assume a constant returns to scale generalized Leontief production function

\[
f(x) = \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} \sqrt{x_i} \sqrt{x_j}
\]

and static competitive profit maximization. This implies

\[
\frac{f(x_1)}{f(x_0)} = \frac{\sum_{i=1}^{N} s_i (x_i,1/x_i,0)^{1/2}}{\sum_{j=1}^{N} s_j (x_j,0/x_j,1)^{1/2}}
\]

where \( s_i \equiv \frac{1}{2} \left( \frac{p_{i,1} x_{i,1}}{\sum_{j=1}^{N} p_{j,1} x_{j,1}} + \frac{p_{i,0} x_{i,0}}{\sum_{j=1}^{N} p_{j,0} x_{j,0}} \right) \)

i.e. the quantity index number \( X_1/X_0 \) corresponding to the right hand side of (8.33) is exact for the production function (8.32).

**Proof.** Competitive profit maximization and constant returns to scale imply \( p \frac{\partial f(x)}{\partial x_i} = w_i \) and \( p f(x) = \sum_{i=1}^{N} w_i x_i \), so that

\[
u_{i,t} \equiv \frac{w_{i,t}}{\sum_{j=1}^{N} w_{j,t} x_{j,t}} = \frac{\partial f(x_t)}{\partial x_i} \cdot \frac{1}{f(x_t)} \quad (i = 1, \cdots, N)
\]


for all time periods \( t \). Substituting the derivatives of (8.32) into (8.34),

\[
v_{i,t} = \frac{(x_{i,t})^{-1/2} \sum_{j=1}^{N} a_{ij} \sqrt{x_{j,t}}}{f(x_{i})}
\]  

(8.35)
multiplying \( v_{i,0} \) by \( \sqrt{x_{i,1}}/\sqrt{x_{i,0}} \) and summing over \( i = 1, \ldots, N \),

\[
\sum_{i=1}^{N} \sqrt{x_{i,1}} \cdot v_{i,0} \cdot \sqrt{x_{i,0}} = \frac{\sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} \cdot \sqrt{x_{i,1}} \cdot \sqrt{x_{j,0}}}{f(x_{0})}
\]  

(8.36)
and similarly,

\[
\sum_{i=1}^{N} \sqrt{x_{i,0}} \cdot v_{i,1} \cdot \sqrt{x_{i,1}} = \frac{\sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} \cdot \sqrt{x_{i,0}} \cdot \sqrt{x_{j,1}}}{f(x_{1})}
\]  

(8.37)
Dividing (8.36) by (8.37) (noting \( a_{i,j} = a_{j,i} \) and \( \sqrt{x_{i,1}}/\sqrt{x_{i,0}} = \sqrt{x_{i,1}/x_{i,0}} \cdot x_{i,0} \)),

\[
\frac{\sum_{i} (x_{i,1}/x_{i,0})^{1/2} \cdot v_{i,0} \cdot x_{i,0}}{\sum_{j} (x_{j,0}/x_{j,1})^{1/2} \cdot v_{j,1} \cdot x_{j,1}} = \frac{f(x_{1})}{f(x_{0})}.
\]  

(8.38)
Alternatively, assume constant returns to scale in production and a Generalized Leontief unit cost function \( c(w) = C(w, y)/y' \):

\[
c(w) = \sum_{i=1}^{N} a_{ij} \sqrt{w_{i}} \sqrt{w_{j}}
\]  

(8.39)
These assumptions and competitive profit maximization imply

\[
c(w_{1}) = \frac{\sum_{i=1}^{N} s_{i,0} \left( w_{i,1}/w_{i,0} \right)^{1/2}}{\sum_{j=1}^{N} s_{j,1} \left( w_{j,0}/w_{j,1} \right)^{1/2}}
\]  

(8.40)
where \( s_{0}, s_{1} \) are defined as in (8.33) (the proof is analogous to the above proof of (8.33)). Thus the above price index for inputs is exact for a Generalized Leontief unit cost function (8.39).

### 8.4 Two-stage aggregation with superlative index numbers

All of the above index number formulas and their relations to Translog and Generalized Leontief functional forms were essentially calculated as a one stage aggregation procedure. For example all inputs \( 1, \ldots, N \) were aggregated directly into a single input quantity index and a single input price index rather than into several quantity
and price subindexes. On the other hand, commodities typically are aggregated into various subindexes for use in econometric models.

This leads to the following important question: are two stage aggregation produces (using index numbers formulas) exact or approximately exact? For example, suppose that the Törnqvist quantity index number formula (8.6) and the corresponding implicit price index formula are applied separately to two subsets of inputs ($N_A$ and $N_B$ inputs, respectively, where $N_A + N_B = N$, the total number of inputs), resulting in two quantity indexes $Q^A, Q^B$

$$Q^A = \log \left( \frac{X_1^A}{X_0^A} \right)$$
$$Q^B = \log \left( \frac{X_1^B}{X_0^B} \right)$$

and corresponding implicit prices indexes $\tilde{p}^A, \tilde{p}^B$. Then the same quantity index number formula (8.6) is applied to the subindexes $Q^A, Q^B, \tilde{p}^A, \tilde{p}^B$, resulting in a quantity index $\hat{Q}$:

$$\hat{Q} = \log \left( \frac{X_1}{X_0} \right)$$

$$= \frac{1}{2} \left( \frac{\tilde{p}^A Q_1^A}{\tilde{p}_1^A Q_1^A + \tilde{p}_0^A Q_0^A} + \frac{\tilde{p}^B Q_1^B}{\tilde{p}_1^B Q_1^B + \tilde{p}_0^B Q_0^B} \right) \log \left( \frac{Q_1^A}{Q_0^A} \right)$$

$$+ \frac{1}{2} \left( \frac{\tilde{p}^A Q_0^A}{\tilde{p}_1^A Q_1^A + \tilde{p}_0^A Q_0^A} + \frac{\tilde{p}^B Q_0^B}{\tilde{p}_1^B Q_1^B + \tilde{p}_0^B Q_0^B} \right) \log \left( \frac{Q_0^A}{Q_0^B} \right)$$

Are the results of one stage and two stage aggregation identical? For particular, is the quantity index $\hat{Q}$ calculated in (8.41)–(8.42) exact for a constant returns to scale Translog production function $y = f(x^1, \ldots, x^N)$, i.e. does $\hat{Q}$ equal $\log(f(x_1)/f(x_0))$?

In general the two stage aggregation procedure (8.41)–(8.42) is not exact, i.e. $\hat{Q} \neq \log(f(x_1) / f(x_0))$. This result is not surprising, since results in the previous chapter indicated that two stage budgeting is correct only under strong restrictions on the structure of production or utility functions (e.g. homothetic weak separability).

On the other hand the two stage aggregation procedure (8.41)–(8.42) is approximately exact, i.e. $\hat{Q}$ approximates $\log(f(x_1)/f(x_0))$. Moreover, two stage aggregation procedures based on known superlative index number formulas are approximately exact. This can be explained very briefly as follows (see Diewert 1978).
The Vartia price index $P^V$ and quantity index $Q^V$, where

$$\log P^V = \sum_{i=1}^{N} s_i \log \left( \frac{P_{i,1}}{P_{i,0}} \right)$$

$$\log Q^V = \sum_{i=1}^{N} s_i \log \left( \frac{Q_{i,1}}{Q_{i,0}} \right)$$

where $s_i = \left[ \frac{p_{i,1}x_{i,1} - p_{i,0}x_{i,0}}{\log \left( \frac{p_{i,1}x_{i,1}}{p_{i,0}x_{i,0}} \right)} \right] / \left[ \frac{\sum_{j=1}^{N} p_{j,1}x_{j,1} - \sum_{j=1}^{N} p_{j,0}x_{j,0}}{\log \left( \frac{\sum_{j=1}^{N} p_{j,1}x_{j,1}}{\sum_{j=1}^{N} p_{j,0}x_{j,0}} \right)} \right]$, (8.43)

are known to be consistent in aggregation, i.e., one stage and two stage aggregation procedures based on (8.43) lead to identical index numbers. Unfortunately these Vartia indexes are exact for a constant returns to scale Cobb-Douglas production function $y = f(x_1, \ldots, x_N)$ rather than for a second order flexible functional form.\(^4\)

Nevertheless it can be shown that the Vartia quantity index $Q^V$ differentially approximates the Törnqvist quantity index $Q^T \equiv (X_1/X_0)$ (8.6) to the second order at any point where the prices and quantities are the same for the two time periods comparison ($p_1 = p_0, x_1 = x_0$), i.e.

$$\frac{Q^V(z)}{\partial z_i} = \frac{Q^T(z)}{\partial z_i}$$

$$\frac{\partial^2 Q^V(z)}{\partial z_i \partial z_j} = \frac{\partial^2 Q^T(z)}{\partial z_i \partial z_j}$$

for all $i, j$ (8.44)

where $z \equiv (p_1, p_0, x_1, x_0)$ and $p_1 = p_0, x_1 = x_0$. Identical results hold for the Vartia price index $P^V$ and the Törnqvist price index $p^T \equiv (W_1/W_0)$ (8.18).

These results do not require any assumptions about optimizing behavior log agents. Moreover, these results can be extended to other superlative indexes, e.g., indexes corresponding to Generalized Leontief production functions or cost functions.

Therefore two stage aggregation procedures using superlative index number formulas are approximately correct (exact) for relatively small changes in prices and quantities between the comparison time periods $t = 0, 1$. These changes in prices and quantities are usually smaller when indexes are constructed by chaining observations in successive period rather than using a constant base period. On this basis it is recommended that a Törnqvist quantity macroindex or subindex (8.6), e.g., be constructed as

---

\(^4\) Laspeyres and Paasche index numbers are also consistent in aggregation, but the corresponding production are much more restrictive than the Cobb-Douglas (see Section 8.1).
8.5 Conclusion

The above results indicate that, at least in the case of simple static maximization models, various index number procedures can be used to obtain approximately consistent aggregation of commodities provided that the variation in prices and quantities between comparison periods is small. For time series data this condition can usually be satisfied by chaining observations in successive periods.

However in the case of cross-section data there may be substantial variation in quantities or prices between successive periods, and here different superlative index number formulas may lead to significantly different results. In this case it may be useful to compare the variation in the $N$ quantity ratios $x_{i,1}/x_{i,0}$ to the variation in the $N$ price ratios $p_{i,1}/p_{i,0}$.

Usually there is much less variation in the price ratios than in the quantity ratios. Then a directly defined price index $P^A_t = \sum_{i=1}^{N_A} s^A_{i,t} \log\left(\frac{p^A_{i,t}}{p^A_{i,t-1}}\right)$ is less sensitive to the variation in data than is a directly defined quantity index $\sum_{i=1}^{N_A} s^A_{i,t} \log\left(\frac{x^A_{i,t}}{x^A_{i,t-1}}\right)$ (since both equations use the same shares $s^A_{i,t}$). This suggests that the best strategy in this case is to calculate the price indexes directly and to employ the corresponding implicit quantity indexes (see Allen 1981, pp. 430-435).

References


8.6 Laspeyres and Paasche cost of living indexes

\[
\begin{align*}
\min_{\mathbf{x}} & \quad \mathbf{p} \mathbf{x} \\
\text{s.t.} & \quad u(\mathbf{x}) = u^* \rightarrow E(\mathbf{p}, u)
\end{align*}
\]

\(u(\mathbf{x})\) homothetic \(E(\mathbf{p}, u) = u E(\mathbf{p}, 1)\).

True cost of living (COL) index conditional on \(u^*\):

\[
\frac{P_1}{P_2} = \frac{E(p_1, u^*)}{E(p_0, u^*)}
\]

Homotheticity \(\Rightarrow\) COL index is independent of any \(u^*\):

\[
\frac{P_1}{P_0} = \frac{u^* \cdot e(p_1)}{u^* \cdot e(p_0)} = \frac{e(p_1)}{e(p_0)}.
\]

Laspeyres COL index is \(>\) true COL (at \(u_0^*\)):

\[
\left(\frac{P_1}{P_0}\right)^L \equiv \frac{x_0 p_1}{x_0 p_0} > \frac{E(p_1, u_0^*)}{E(p_0, u_0^*)}.
\]

Paasche COL index is \(<\) true COL (at \(u_1^*\)):

\[
\left(\frac{P_1}{P_0}\right)^P \equiv \frac{x_1 p_1}{x_1 p_0} > \frac{E(p_1, u_1^*)}{E(p_0, u_1^*)}.
\]

So (assuming homotheticity)

\[
\left(\frac{P_1}{P_0}\right)^P < \text{true COL} < \left(\frac{P_1}{P_0}\right)^L \quad (8.47)
\]

Note: (8.47) marks no assumptions about form of preferences (such as Translog), except for homotheticity.

If (8.47) does no place close enough bounds on true COL, then we should specify (e.g.,) a Törnqvist consumer price index (assuming a Translog CRTS cost function \(E(p, u)\)).
8.7 Fisher indexes (for inputs)

CRTS → C(w, y) = yC(w, 1) where C(w, 1) ≡ c(w): unit cost function.
Assume the following quadratic c(w) function:
\[
c(w) = \left[ \sum_i \sum_j a_{ij} w_i w_j \right]^{1/2}
\] (8.48)

Define the following price index:
\[
\left( \frac{W_1}{W_0} \right)^F = \left[ \left( \frac{W_1}{W_0} \right)^L \left( \frac{W_1}{W_0} \right)^P \right]^{1/2}
\] (8.49)

This is called a Fisher price index (it is a geometric mean of a Laspeyres and Paasche index).

We can prove the following result:

**Theorem 8.1.** Assume CRTS and the quadratic cost function (8.48) (all inputs are at static cost minimizing equilibrium). Then
\[
\left( \frac{W_1}{W_0} \right)^F = \frac{c(w_1)}{c(w_0)} = \frac{AC_1}{AC_0}.
\]

Alternatively define a Fisher input quantity index as
\[
\left( \frac{X_1}{X_0} \right)^F = \left[ \left( \frac{X_1}{X_0} \right)^L \left( \frac{X_1}{X_0} \right)^P \right]^{1/2}
\] (8.50)

and assume a quadratic CRTS production function
\[
y = \left[ \sum_i \sum_j b_{ij} x_i x_j \right]^{1/2}
\] (8.51)

Then, assuming all inputs are at static cost minimizing equilibrium, we can show
\[
\left( \frac{X_1}{X_0} \right)^F = \frac{y_1}{y_0}.
\]

Note: A quadratic cost function (8.48) is essentially equivalent to a quadratic production function (8.51). (In contrast, Translog and G.L. forms are not “self-dual”).
Proof. Assuming CRTS, define the unit cost function \( c(w) = C(w, y) / y \). Assume
\[
c(w) = \left[ \sum_i \sum_j a_{ij} w_i w_j \right]^{1/2}
\]
(8.52)
So
\[
\frac{\partial c(w)}{\partial w_i} = \frac{1}{2} \left[ \sum_i \sum_j a_{ij} w_i w_j \right]^{-1/2} 2 \sum_j a_{ij} w_j \quad (a_{ij} = a_{ji})
\]
(8.53)
So
\[
\frac{\partial c(w)}{\partial w_i} \frac{c(w)}{c(w) c(w)} = \frac{\sum_j a_{ij} w_j}{c(w)}
\]
by (8.52)
Cost minimization for all inputs implies Shephard’s Lemma
\[
y \frac{\partial c(w)}{\partial w_i} = x_i.
\]
(8.55)
Dividing (8.55) by \( y c(w) \equiv \sum_j w_j x_j \),
\[
\frac{x_i}{\sum_j w_j x_j} = \frac{\partial c(w)}{\partial w_i} \frac{c(w)}{c(w)}
\]
(8.56)
The Fisher input price index is
\[
\left( \frac{W_1}{W_0} \right)^F = \left[ (W_1 / W_0)^L (W_1 / W_0)^P \right]^{1/2}
\]
\[
= \left[ (w_1 x_0 / w_0 x_0) \cdot (w_1 x_1 / w_0 x_1) \right]^{1/2}
\]
\[
= \left[ (w_1 x_0 / w_0 x_0) / (w_0 x_1 / w_1 x_1) \right]^{1/2}
\]
\[
= \left[ \left( w_1 \frac{\partial c(w_0)}{\partial w} \right) / \left( w_0 \frac{\partial c(w_1)}{\partial w} \right) \right]^{1/2}
\]
by (8.56)
\[
= \left[ \sum_i \sum_j \frac{w_{i,1} w_j,0}{c(w_0)^2} / \sum_i \sum_j \frac{w_{i,0} w_{i,1}}{c(w_1)^2} \right]^{1/2}
\]
by (8.54)
\[
= \left[ 1/c(w_0)^2 / 1/c(w_1)^2 \right]^{1/2}
\]
\[
= c(w_1) / c(w_0)
\]
In previous chapters we assumed the existence of an unchanged production function \( y = f(x) \) as transformation function \( g(y, x) = 0 \), where output levels \( y \) are determined by input levels \( x \). In contrast here we allow for shifts in the technology of the firm or industry that lead to changes in output levels \( y \) that cannot be accounted for solely by changes in input levels \( x \).

In principle studies of productivity as in this chapter control for changes in the quality of measured inputs. Consequently application of these methods to a particular sector of the economy (e.g. agriculture) will not (in principle) quantify the effect of improved quality of measured inputs on the sector’s output. Improvement in quality of inputs should be analyzed as technical changes in the industries producing those factors. In order to study the effect of improvements in quality of measured inputs as well as changes in levels of unmeasured inputs on agricultural output, it has been suggested that agriculture and manufacturing industries supplying measured inputs to agriculture should be combined into a single sector where these inputs are treated as intermediate goods (Kisler and Peterson 1981).

This chapter introduces several approaches to the measurement of technical change. Section 9.1 considers econometric production models incorporating a time tread as a proxy for technical change. Section 9.2 summarizes index number procedures for calculating changes in productivity without recourse to econometrics or specification of specific functional forms for the production function, cost function or profit function (in this sense the procedures are non-parametric). Section 9.3 summarizes parametric index number procedures for calculating changes in productivity without recourse to econometrics. These sections 9.2 and 9.3 both assume that there is no variation in utilization rates of capital over time; i.e. the industry or firm is assumed to be in long-run equilibrium. Section 9.4 demonstrates that (under re-
strictive assumptions) these non-econometric procedures can be modified to allow for variable utilization rates in capital, and that econometric procedures can (under certain restrictive assumptions) allow for variable utilization rates in capital. Section 9.5 conclude the chapter.

9.1 Dual cost and profit functions with technical change

In both primal and dual econometric models of the industry or firm, technical changes usually is proxied simply by a time trend variable $t = 1, 2, 3, \cdots, T$ where $T$ is the number of time periods. By allowing for interactions between this time trend and other variables in the model, this specification is designed to proxy more than a regular secular time trend for technical changes over historical time. Of course the time trend may also proxy secular trends in any relevant variables that have been excluded from the model. In addition it is assumed that technical change is exogenous to the industry or firm.

A multiple output Translog cost function $C = C(w, y)$ can be generalized to incorporate a time trend $t = 1, 2, 3, \cdots$, as follows:

$$
\log C = \alpha_0 + \sum_{i=1}^{N} \alpha_i \log w_i + \sum_{i=1}^{M} \beta_i \log y_i + \varphi_1 t \\
+ \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_{ij} \log w_i \log w_j \\
+ \frac{1}{2} \sum_{i=1}^{M} \sum_{j=1}^{M} \beta_{ij} \log y_i \log y_j \\
+ \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{M} \gamma_{ij} \log w_i \log y_j \\
+ \sum_{i=1}^{N} \phi_i t \log w_i + \sum_{i=1}^{M} \delta_i t \log y_i + \varphi_2 t^2
$$

(9.1)

Shephard’s lemma implies the following factor share equations:

$$
s_i \equiv \frac{w_i x_i}{C} = \alpha_i + \sum_{j=1}^{N} \alpha_{ij} \log w_j + \sum_{j=1}^{M} \gamma_{ij} \log y_j + \phi_i t \quad i = 1, \cdots, N
$$

(9.2)

Assuming competitive profit maximization, the first order conditions $p_j = \frac{\partial C(w, y, t)}{\partial y_j}$ ($j = 1, \cdots, M$) and (9.1) imply
\[ p_j y_j \quad 9.1 \text{ Dual cost and profit functions with technical change} \quad \frac{\partial \log C}{\partial \log y_j} \]

\[ = \beta_j + \sum_{i=1}^{N} \gamma_{ij} \log w_i + \sum_{j=1}^{M} \beta_{ji} \log y_j + \delta_j t \quad j = 1, \ldots, M \quad (9.3) \]

Differentiating (9.1) with respect to \( t \),

\[ \frac{\partial \log C}{\partial t} = \frac{\partial C(w, y, t)}{C(w, y, t)} \frac{\partial}{\partial t} \]

\[ = \varphi_1 + \sum_{i=1}^{N} \varphi_i \log w_i + \sum_{j=1}^{M} \delta_j \log y_j + 2\varphi_2 t \quad (9.4) \]

Assuming technical progress (not regress), \( \frac{\partial \log C}{\partial t} < 0 \). Note that equation (9.4) cannot be estimated directly because the percentage reduction in cost due to technical change \( \frac{\partial \log C}{\partial t} \) is unobserved, and also note that the coefficients \( \varphi_1, \varphi_2 \) of (9.4) cannot be inferred from estimates of factor demand and output supply equation (9.2)–(9.3). In order to obtain estimates of all coefficients of (9.4), it is necessary to estimate directly equation (9.1) defining the Translog cost function. Unfortunately the data will often permit estimation of equation (9.2)–(9.3), but not direct estimation of the cost function equation (9.1).

Nevertheless we can easily derive a measure of technical change \( \frac{\partial C(w, y, t)}{\partial t} \) directly from estimates of the factor share equation (9.2). Since \( C(w, y, t) = \sum_{i=1}^{N} x_i(w, y, t)w_i \),

\[ \frac{\partial C(w, y, t)}{\partial t} = \sum_{i=1}^{N} w_i \frac{\partial x_i(w, y, t)}{\partial t} \quad (9.5) \]

Differentiating \( s_i(w, y, t) \equiv w_i x_i(w, y, t) / C(w, y, t) \) with respect to \( t \) and substituting in (9.5) yields \( \frac{\partial C}{\partial t} = A \frac{\partial x}{\partial t} \) where matrix \( A \) has full rank. Then \( \frac{\partial x}{\partial t} \) can be calculated as \( \frac{\partial x}{\partial t} = A^{-1} \phi \) using the estimates of \( \frac{\partial x}{\partial t} \) from the share equations (9.2), and in turn \( \frac{\partial C}{\partial t} \) can be calculated using (9.5).

A single output generalized Leontief cost function with technical change can be specified as

\[ C = y \sum_{i} \sum_{j} a_{ij} \sqrt{w_i} \sqrt{w_j} + y^2 \sum_{i} a_i w_i + ty \sum_{i} b_i w_i \quad (9.6) \]

\[ A = \begin{bmatrix} \frac{w_1}{c} (1 - \frac{w_1}{c}) & -\frac{w_1}{c} \frac{w_2}{c} & \cdots & -\frac{w_1}{c} \frac{w_N}{c} \\ -\frac{w_1}{c} \frac{w_1}{c} & \frac{w_2}{c} (1 - \frac{w_2}{c}) & \cdots & -\frac{w_2}{c} \frac{w_N}{c} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{w_1}{c} \frac{w_1}{c} & -\frac{w_2}{c} \frac{w_2}{c} & \cdots & \frac{w_N}{c} (1 - \frac{w_N}{c}) \end{bmatrix} \]

which generally has full rank. Deleting the equation for share \( s_N \) from the estimation of (9.2), \( \phi_N = 1 - \sum_{i=1}^{N-1} \phi_i \).
Here $\frac{\partial C(w, y, t)}{\partial t} = y \sum_{i=1}^{N} b_i w_i$, and all coefficients of this equation can be recovered directly from the estimated factor demand equations (in contrast to the Translog).

The shift in the underlying production function or transformation function due to technical change can easily be calculated from the reduction in cost $\frac{\partial C(w, y, t)}{\partial t}$. Assuming competitive cost minimization and a production function $y = f(x, t)$ for a single output $y$, the increase in output $\frac{\partial f(x^*, t)}{\partial t}$ due to technical change can be calculated as follows:

$$\frac{\partial f(x^*, t)}{\partial t} = -\frac{\partial C(w, y, t)}{\partial t} / \frac{\partial C(w, y, t)}{\partial y}$$  \hspace{1cm} (9.7)

or, assuming competitive profit maximization,

$$\frac{\partial f(x^*, t)}{\partial t} = -\frac{\partial C(w, y, t)}{\partial t} / p$$  \hspace{1cm} (9.8)

In the case of constant returns to scale in production $C(w, y, t) = C_y(w, y, t)y$ (Euler’s theorem), and in turn (9.7) implies

$$\frac{\partial f(x^*, t)}{\partial t} / y = -\frac{\partial C(w, y, t)}{\partial t} / C$$  \hspace{1cm} (9.9)

Thus under constant returns to scale the rate of growth of output (at equilibrium $x^*$) is equal to the rate of reduction in cost due to technical change. Similarly in the case of a multiple output transformation function $y_1 = g(y_2, \cdots, y_M, x, t) \equiv y_1(\tilde{y}, x, t)$ and assuming competitive profit maximization, the shift in transformation function due to technical change $\frac{\partial g(\tilde{y}, x, t)}{\partial t}$ can be calculated as

$$\frac{\partial g(\tilde{y}, x, t)}{\partial t} = -\frac{\partial C(w, y, t)}{\partial t} / p_1$$  \hspace{1cm} (9.10)

These relations (9.7)-(9.10) are proved in the next section 9.2.

Biases as well as magnitudes of technical change can be calculated from cost functions. For this purpose it is convenient to define Hicks-neutral technical change as technical change that does not alter the firm or industry’s input expansion path. Then the change in cost shares $\frac{\partial s_i(w, y, t)}{\partial t} (i = 1, \cdots, N)$ provides a measure of bias if the production function is homothetic, i.e. the expansion in input space is linear from the origin for a given state of technology.

However the change in shares $\frac{\partial s_i(w, y, t)}{\partial t}$ does not provide an accurate measure of biases in technical change when the production function is non-homothetic. For example suppose that Hicks-neutral technical change occurs, i.e. technical change does not alter the input expansion path. This technical change simply leads to a re-numbering of the output levels for given isoquants ($\frac{\partial f(x^*, t)}{\partial t} > 0$ and output $y$ constant implies the firm moves to a lower isoquant). If the expansion path is not linear and the firm’s output level must be held constant as this reindexing occurs, then the firm moves to a different isoquant with a different cost-minimizing
ratio of inputs. Thus the changes in shares $\frac{\partial s_i(w, y, t)}{\partial t}$ $(i = 1, \ldots, N)$ are not equal to zero even though technical change is Hicks-neutral.

In order to correct for non-homotheticity in calculating biases in technical change, note that the change in shares can be decomposed into a scale effect due to movement along the initial expansion path and a bias effect due to the shift in the expansion path. The share equation $s_i = s_i(w, i^*(t), t)$ can be defined equivalently as $s_i = \dot{s}_i(w, i^*(t), t)$ where $i^*(t)$ indexes the isoquant yielding output level $y$ given technology $t$. Differentiation of this composite function with respect to $t$ yields

\[
\frac{\partial s_i(w, y, t)}{\partial t} = \frac{\partial \dot{s}_i(w, i^*(t), t)}{\partial t} + \frac{\partial s_i(w, y, t)}{\partial y} \frac{\partial f(x^*, t)}{\partial t}
\]

(9.11)

where $\frac{\partial \dot{s}_i(w, i^*(t), t)}{\partial t}$ provides a true measure of bias in technical change. Combining (9.7) and (9.11), a true measure of bias in technical change for a non-homothetic production function can be defined in terms of a cost function as

\[
\frac{\partial s_i(w, y, t)}{\partial t} + \frac{\partial s_i(w, y, t)}{\partial y} \frac{\partial C(w, y, t)}{\partial t} / \frac{\partial C(w, y, t)}{\partial y} \quad i = 1, \ldots, N
\]

(9.12)

or in elasticity terms as

\[
\frac{\partial \log s_i(w, y, t)}{\partial t} + \frac{\partial \log s_i(w, y, t)}{\partial \log y} \frac{\partial \log C(w, y, t)}{\partial t} / \frac{\partial \log C(w, y, t)}{\partial \log y} \quad i = 1, \ldots, N
\]

(9.13)

Homotheticity implies $\frac{\partial \dot{s}_i(w, x, t)}{\partial y} = \frac{\partial \log s_i(w, y, t)}{\partial \log y} = 0$ $(i = 1, \ldots, N)$. Similarly in the case of multiple outputs, a true measure of bias in technical change can be calculated as

\[
\frac{\partial \log s_i(w, y, t)}{\partial t} + \sum_{j=1}^{M} \frac{\partial \log s_i(w, y, t)}{\partial \log y_j} \frac{\partial \log C(w, y, t)}{\partial t} / \frac{\partial \log C(w, y, t)}{\partial \log y_j}
\]

(9.14)

Technical change can be incorporated into dual profit functions in a manner analogous to the above treatment of cost functions. Here we simply note the following relation between changes in profits and costs due to technical change

\[
\frac{\partial \pi(w, p, t)}{\partial t} = -\frac{\partial C(w, y^*, t)}{\partial t}
\]

(9.15)

This relation is proved in the next section 9.2.

Finally it is important to note that most of the above discussion applies equally well to short-run and long-run equilibrium models. The above models assumed that all inputs are freely variable and attain static equilibrium levels. In contrast we could postulate, e.g., a Translog cost function $C(w, y, k, t)$ conditional on the stocks $K$ of quasi-fixed inputs.

In place of (9.7) we have the following relation between the shift $\frac{\partial f(x^*, K, t)}{\partial t}$ in the production function and the shift $\frac{\partial C(w, y, k, t)}{\partial t}$ in cost:
where \( C_y(w, y, K, t) = p \) assuming competitive profit maximization. On the other hand the measurement of biases is complicated somewhat when the cost function is short-run equilibrium in nature.

### 9.2 Non-Parametric index number calculations of changes in productivity

In this section we consider methods for calculating technical change that require neither econometrics nor the specification of particular functional forms for a production function as cost function. However this index number approach to productivity is defined in terms of continuous time, and errors in approximation result from the use of discrete time data. Moreover these calculations usually assume that all inputs are freely variable and are at static long-run equilibrium levels.

Initially assume a single output production function \( y = f(x(t)). \) Then output at time \( t \) is related to inputs at time \( t \) and technology as indicated by the relation \( y(t) = f(x(t), t). \) Differentiating with respect to \( t \) yields

\[
\frac{\partial f(x^*, K, t)}{\partial t} = -\frac{\partial C(w, y, K, t)}{\partial t} \cdot \frac{\partial C(w, y, K, t)}{\partial y} \tag{9.16}
\]

The weightings \( w_i x_i = py \) for input changes \( \hat{x}_i/x_i \) define a Divisia quantity index for inputs in continuous time. Thus, assuming continuous time and all inputs are at

\[
\hat{y} = \sum_{i=1}^{N} \left( \frac{w_i^i}{C_y(w, y, t)} \right) \hat{x}_i^i + \frac{\partial f(x, t)}{\partial t} \tag{9.17}
\]

where \( \hat{y} \equiv \frac{\partial y(t)}{\partial t}, \hat{x} \equiv \frac{\partial x(t)}{\partial t} \) denotes the change in output and input levels with respect to time. The standard first order conditions for static competitive cost minimization are \( w_i - C_y(w, y, t) \frac{\partial f(x^*, t)}{\partial x_i^i} = 0 \) \( (i = 1, \ldots, N) \), and substituting these into (9.17) yields

\[
\hat{y} = \sum_{i=1}^{N} \left( \frac{w_i^i}{C_y(w, y, t)} \right) \hat{x}_i^i + \frac{\partial f(x, t)}{\partial t} \tag{9.18}
\]

Competitive profit maximization implies \( C_y(w, y, t) = p \), and substituting this into (9.18) yields

\[
\hat{y} = \sum_{i=1}^{N} \left( \frac{w_i^i}{p} \right) \hat{x}_i^i + \frac{\partial f(x, t)}{\partial t} \tag{9.19}
\]

or equivalently

\[
\frac{\hat{y}}{y} = \sum_{i=1}^{N} \left( \frac{w_i^i x_i}{p y} \right) \frac{\hat{x}_i}{x_i} + \frac{\partial f(x, t)}{\partial t} / y \tag{9.20}
\]
9.2 Non-Parametric index number calculations of changes in productivity

Static long-run equilibrium levels, technical change ∂f(x, t)/∂t could be calculated as the residual in (9.18) or (9.19)-(9.20).

However data is measured at discrete time intervals rather than continuously, and equation (9.19) or (9.20) can only be approximated using discrete data. For example integrating (9.19) over an interval \( t = 0, 1 \) yields

\[
y(1) - y(0) = \sum_{i=1}^{N} \int_{t=0}^{1} \frac{w^i}{p} \frac{dx^i(t)}{dt} dt + \int_{t=0}^{1} \frac{\partial f(x(t), t)}{\partial t} dt \quad (9.21)
\]

However, unless all price ratios \( w^i/p \) are independent of \( t \) \( (i = 1, \ldots, N) \), closed form solutions for the integrals \( \int_{t=0}^{1} \frac{w^i}{p} \frac{dx^i(t)}{dt} dt \) generally cannot be defined. (9.20) is most commonly approximated using Törnqvist approximations to the Divisia index:

\[
T_F \approx \frac{y(t) - y(t - 1)}{y(t)} - \sum_{i=1}^{N} \frac{1}{2} \left( \frac{w^i}{p^y} \frac{dx^i}{dy}(t) + \frac{w^i}{p^y} (x^i(t) - x^i(t - 1)) \right) \frac{x^i(t) - x^i(t - 1)}{x^i(t)} \quad (9.22)
\]

where \( T_F \equiv \int_{v=t-1}^{t} \frac{\partial f(x(v), v)}{\partial v} dy(v)dv \) denotes the integral of the residual in (9.19) attributed to technical change.

There is a further complication in interpreting the calculated shifts in the production function \( \partial f(x^*, t)/\partial t \). The equilibrium point \( x^* \) where the continuous on discrete calculations are made varies over time. Therefore, in order to interpret these results as simple comparable indicators of shifts in the production function over time, it is necessary to assume Hicks-neutral technical change and constant returns to scale. Similar comments apply to the interpretation of other index number calculations of productivity presented in this section and the following section C.

In the case of multiple outputs \( y = (y^1, \ldots, y^M) \), we can define a transformation function normalized on \( y^1 \): \( y^1 = g(\tilde{y}, x, t) \) where \( \tilde{y} \equiv (y^2, \ldots, y^M) \). Differentiating \( y^1(t) = g(\tilde{y}(t), x(t), t) \) with respect to \( t \) yields

\[
\dot{y}^1 = \sum_{i=2}^{M} \dot{g}(\tilde{y}, x, t) \dot{\tilde{y}}^i + \sum_{i=1}^{N} g_{x^i}(\tilde{y}, x, t) \dot{x}^i + \partial g(\tilde{y}, x, t)/\partial t \quad (9.23)
\]

The competitive maximization problem \( \max_{x, \tilde{y}, \lambda} L \equiv py - wx - \lambda(g(\tilde{y}, x, t) - y^1) \) has first order conditions \( p' + p^1 g_{y^i}(\tilde{y}, x, t) = 0 \) \( (i = 2, \ldots, M), -w^i p' g_{x^i}(\tilde{y}, x, t) = 0 \) \( (i = 1, \ldots, N) \), and substituting these into (9.23) yields

\[
\sum_{i=1}^{M} (p^i/p^1) \dot{\tilde{y}}^i = \sum_{i=1}^{N} (w^i/p^1) \dot{x}^i + \partial g(\tilde{y}, x, t)/\partial t \quad (9.24)
\]

However the numerical measure of technical change \( \partial g(\tilde{y}, x, t)/\partial t \) varies with the choice of output \( i \), i.e. the measure of technical change is not symmetric with respect to the normalization of the transformation function.
In the case of multiple outputs a move useful measure of technical change may be obtained in terms of the cost function \( C(w, y, t) \). At time \( t \) \( C(w(t), y(t), t) = \sum_{i=1}^{N} w^i(t)x^i(t) \), and differentiating with respect to \( t \) yields

\[
\sum_{i=1}^{N} C_{w^i}(w, y, t)\dot{w}^i + \sum_{j=1}^{M} C_{y^j}(w, y, t)\dot{y}^j + \frac{\partial C(w, y, t)}{\partial t} = \sum_{i=1}^{N} \dot{w}^i x^i + \sum_{i=1}^{N} w^i \dot{x}^i
\]  

(9.25)

Competitive cost minimization implies \( C_{w^i}(w, y, t) = x^i(w, y, t) \) \( (i = 2, \cdots, N) \) (Shephard’s lemma), and substituting these conditions into (9.25) yields

\[
\frac{\partial C(w, y, t)}{\partial t} = \sum_{i=1}^{N} w^i \dot{x}^i - \sum_{j=1}^{M} C_{y^j}(w, y, t)\dot{y}^j \]  

(9.26)

Competitive profit maximization further implies \( C_{y^j}(w, y, t) = p^j \) \( (j = 1, \cdots, M) \), and substituting into (9.26) yields

\[
\frac{\partial C(w, y, t)}{\partial t} = \sum_{i=1}^{N} w^i \dot{x}^i - \sum_{j=1}^{M} p^j \dot{y}^j \]

(9.27)

or equivalently

\[
\frac{\partial C(w, y, t)}{\partial t} = \sum_{i=1}^{N} \left( \frac{w^i x^i}{C} \right) \dot{x}^i - \sum_{j=1}^{M} \left( \frac{p^j y_j}{C} \right) \dot{y}^j \]  

(9.28)

How the simple relations between primal measures and dual cost and profit measures of technical change can easily be established as follows. Comparing (9.18) and (9.26) for the case of a single output and cost minimization establishes

\[
\frac{\partial C(w, y, t)}{\partial t} = -\frac{\partial f(x^*, t)}{\partial t} \frac{\partial C(w, y, t)}{\partial y} \]  

(9.29)

or in the case of competitive profit maximization

\[
\frac{\partial C(w, y, t)}{\partial t} = -p \frac{\partial f(x^*, t)}{\partial t} \]  

(9.30)

Similarly comparing (9.24) and (9.27) in the case of multiple outputs and profit maximization establishes

\[
\frac{\partial C(w, y, t)}{\partial t} = -p \frac{\partial g(y^*, x^*, t)}{\partial t} \]  

(9.31)

In term of the profit function \( \pi(w, p, t) \) we have the identity \( \pi(w(t), p(t), t) = p(t)y(t) - w(t)x(t) \) for profits at time \( t \), and differentiating with respect to \( t \) and applying Hotelling’s lemma yields
Comparing (9.27) and (9.32) establishes

\[ \frac{\partial \pi(w, p, t)}{\partial t} = \sum_{i=1}^{M} p^i \dot{y}^i - \sum_{i=1}^{N} w^i \dot{x}^i \] (9.32)

Comparing (9.27) and (9.32) establishes

\[ \frac{\partial C(w, y^*, t)}{\partial t} = -\frac{\partial \pi(w, p, t)}{\partial t} \] (9.33)

9.3 Parametric index number calculations of changes in productivity

The Divisia approaches of the previous section required the approximation of continuous time derivatives by discrete differences but did not require the specification of a particular functional form for a production function or cost function. In this section we discuss an alternative approach to calculating changes in productivity: a flexible functional form is assumed for a production function or cost function, and then an index number formula is derived that is consistent with the functional form and with discrete time data. Here there are no errors in approximation due to the use of data for discrete time intervals, but there are errors in approximation to the true functional form for the production on cost function. Here we derive index numbers formulas for technical change corresponding to thus different functional forms: a Translog cost function, a Translog transformation function, and a Translog profit function.

First assume a multiple output Translog cost function \( C(w, y, t) \) as in (9.1). The underlying transformation function need not be constant returns to scale. Since this cost function is a quadratic form, the quadratic identity 8.11 of chapter 8 implies
where $T$ is trend variable identical to 8.26 discussed on pages 8.10-8.11 of chapter 8, except that a time trend variable $t$ is assumed to be a competitive profit maximizer with all inputs variable and by Shephard's lemma and 

\[
\frac{\partial C_1}{\partial \log w^i} + \frac{\partial C_0}{\partial \log w^i} \log \left( \frac{w_i^j}{w_0^j} \right)
\]

\[
+ \frac{1}{2} \sum_{i=1}^{M} \left[ \frac{\partial C_1}{\partial y_i} + \frac{\partial C_0}{\partial y_i} \right] \log \left( \frac{y_i^j}{y_0^j} \right)
\]

\[
+ \frac{1}{2} \left[ \frac{\partial C_1}{\partial t} + \frac{\partial C_0}{\partial t} \right] (t_1 - t_0)
\]

\[
= \frac{1}{2} \sum_{i=1}^{N} \left[ \frac{\partial C_1}{\partial w^i} \frac{w_i}{C_1} + \frac{\partial C_0}{\partial w^i} \frac{w_0}{C_0} \right] \log \left( \frac{w_i^j}{w_0^j} \right)
\]

\[
+ \frac{1}{2} \sum_{i=1}^{M} \left[ \frac{\partial C_1}{\partial y_i} \frac{y_i}{C_1} + \frac{\partial C_0}{\partial y_i} \frac{y_0}{C_0} \right] \log \left( \frac{y_i^j}{y_0^j} \right)
\]

\[
+ \frac{1}{2} \left[ \frac{\partial C_1}{\partial t} / C_1 + \frac{\partial C_0}{\partial t} / C_0 \right] \text{ since } t_1 = t_0 + 1
\]

by Shephard's lemma and $\partial C / \partial y_i = p^i (i = 1, \ldots, M)$. This index number equation is identical to 8.26 discussed on pages 8.10-8.11 of chapter 8, except that a time trend variable $t = 1, 2, 3, \ldots$ is incorporated into the Translog cost function. The firm is assumed to be a competitive profit maximizer with all inputs variable and attaining static long-run equilibrium levels.

Rearranging (9.34)

\[
\frac{1}{2} \left( T_{1C}^C + T_{0C}^C \right) = \left\{ \log \left( \frac{C_1}{C_0} \right) - \frac{1}{2} \sum_{i=1}^{N} \left[ \frac{w_i^j}{C_1} + \frac{w_0^j}{C_0} \right] \log \left( \frac{w_i^j}{w_0^j} \right) \right\}
\]

\[
- \frac{1}{2} \sum_{i=1}^{M} \left[ \frac{p_i^j}{C_1} + \frac{p_0^j}{C_0} \right] \log \left( \frac{y_i^j}{y_0^j} \right)
\]

(9.35)

where $T_{1C}^C = \frac{\partial C(w_1^j, y_1^j, t_1)}{C(w_1^j, y_1^j, t_1)}$, $C_V = \sum_{i=1}^{N} w_i^j x_i^j (V = 0, 1)^V$. $\frac{1}{2}(T_{1C}^C + T_{0C}^C)$ is the average percentage reduction in cost due to technical change at time $t = 0, 1$.

The second set of terms within brackets \{\cdots\} on the right hand side of (9.35) is the logarithm of a Törnqvist price index $(W_1/W_0)$ for inputs (see equation 8.18), so the brackets \{\cdots\} enclose an implicit Törnqvist quantity index $(x_1/x_0)$ for input
\[
\left(\frac{x_1}{x_0}\right) = \left(\frac{C_1}{C_0}\right) / \left(\frac{w_1}{w_0}\right) \text{ by the factor reversal equation analogous to } 8.16.
\]
The term on the right hand side of (9.35) that is outside the brackets can be interpreted as the logarithm of a quantity index \((Y_1/Y_0)\) for outputs (constant returns to scale in production would imply \(C = py\) and this index would be equivalent to a Törnqvist quantity index for outputs). Hence the average percentage reduction in cost is the logarithm of the ratio of input and output quantity indexes:

\[
\frac{1}{2} \left( T^C_1 + T^C_0 \right) = \log \left( \frac{(y_1/x_0)}{(y_1/y_0)} \right) \quad (9.36)
\]

This calculated change in cost can easily be related to shifts in the production or transformation function at equilibrium levels of commodities \(x^*, y^*\) using equations (9.7)-(9.10).

Second, assume a constant returns to scale Translog production function \(y = f(x, t)\) and competitive cost minimization with all inputs variable at long-run equilibrium levels. Proceeding as in the proof of 8.8, we obtain

\[
\frac{1}{2} \left( T^V_1 + T^V_0 \right) = \log \left( \frac{(y_1/y_0)}{(x_1/x_0)^T} \right) \quad (9.37)
\]

where \(T^V_1 \equiv \frac{\partial f(x_1, y_1)}{\partial y_1}/y_1 (v = 0, 1)\). Here the average percentage change in productivity is equal to the logarithm of the ratio of output and input quantity indexes:

\[
\frac{1}{2} \left( T^V_1 + T^V_0 \right) = \log \left( \frac{(y_1/y_0)}{(x_1/x_0)^T} \right) \quad (9.38)
\]

where \((x_1/x_0)^T\) is the Törnqvist quantity index for inputs. Alternatively if we assume a Translog production function (which is not necessarily constant returns to scale) and competitive profit maximizing behavior with all inputs variable, then we obtain the closely related index number formula

\[
\frac{1}{2} \left( T^V_1 + T^V_0 \right) = \log \left( \frac{(y_1/y_0)}{(x_1/x_0)^T} \right) \quad (9.39)
\]

Similarly in the case of multiple outputs assume a Translog transformation function \(y^1 = g(y, x, t)\) (\(y \equiv (y^2, \ldots, y^M)\)) and competitive profit maximizing behavior with all inputs variable. Then

\[
\frac{1}{2} \left( T^V_1 + T^V_0 \right) = \log \left( \frac{(y_1/y_0)}{(x_1/x_0)^T} \right) \quad (9.40)
\]

\[
-\frac{1}{2} \sum_{i=1}^{N} \left[ \frac{w_1^i x_1^i}{p_1 y_1} + \frac{w_0^i x_0^i}{p_0 y_0} \right] \log \left( \frac{x_1^i}{x_0^i} \right)
\]
where \( T^g_V \equiv \frac{\partial g(y^g, x^g, t_V)}{\partial t} / y_V \) \( (V = 0, 1) \). This measure of technical change varies in a simple manner with the commodity chosen as numeraire in the transformation function. For example, given the alternative normalization \( y^1 = g(y^2, \cdots, y^M, x, t) \) and \( y^2 = h(y^1, y^3, y^4, \cdots, y^M, x, t) \),

\[
\frac{\partial g(y^2, y^M, x^*, t)}{\partial t} = \frac{(p^2/p^1) \partial h(y^1, y^3, \cdots, y^M, x^*, t)}{\partial t} \tag{9.41}
\]

Third, assume a Translog profit function and competitive profit maximization with all inputs variable at long-run equilibrium levels. The quadratic identity and Hotelling’s lemma establish

\[
\frac{1}{2} (T^\pi_1 + T^\pi_0) = \log \left( \frac{\pi_1}{\pi_0} \right) - \frac{1}{2} \sum_{i=1}^{M} \left[ \frac{p'_1 y'_1}{\pi_1} + \frac{p'_0 y'_0}{\pi_0} \right] \log \left( \frac{p'_1}{p'_0} \right) - \frac{1}{2} \sum_{i=1}^{N} \left[ \frac{w'_1 x'_1}{\pi_1} + \frac{w'_0 x'_0}{\pi_0} \right] \log \left( \frac{w'_1}{w'_0} \right) \tag{9.42}
\]

where \( T^\pi_V \equiv \frac{\partial \pi(w_V, p_V, t_V)}{\partial t} / \pi(w_V, p_V, t_V) \). This measure of the effect of technical change on profits \( \pi(w, p, t) \) is easily related to changes in costs and shifts in the production function or transformation function using equations (9.7)-(9.10). Interpreting the second and third terms of the right hand side of (9.42) as price indexes for outputs and inputs, respectively, and noting that \( \pi \equiv p y - w x \), it follows that \( \frac{1}{2} (T^\pi_1 + T^\pi_0) \) is the logarithm of the ratio of implicit quantity indexes for outputs and for inputs.

The behavioral models of the firm employed in sections B and C have assumed static long-run equilibrium and lead to index number formulas such as (9.20) that are defined in terms of the flows \( x_t \) of all inputs used in production. Since the flow of capital services is not generally observable, it is usually assumed that the flow of capital services is proportional to the stock of capital assets (this assumption may be reasonable at long-run equilibrium). Then the growth rate of the capital service flow \( \frac{\Delta K}{K} \) is equal to the growth rate of the capital stock \( \frac{\Delta K}{K} \).

The capital stock \( K_t \) is often approximated by the perpetual inventory method:

\[
K_t = I_{t-1} + (1-\delta)K_{t-1} \quad \text{where } I_{t-1} \text{ denotes gross investment at time } t-1 \text{ and } \delta \text{ is a constant rate of depreciation, and substituting backwards for } K_{t-1}, K_{t-2}, \cdots, K_{t-S} \text{ yields the approximation}
\]

\[
K_t \approx I_{t-1} + \sum_{S=2}^{S} (1-\delta)^{t-S} I_{t-S} \tag{9.43}
\]

Thus time series data on capital stock \( K \) and in turn the growth rate of the capital service flow is approximated from data on gross investment \( I \) and an assumed rate of depreciation \( \delta \). Then the rate of technical change is calculated using index number formulas such as (9.20), (9.28), (9.35), (9.37) (e.g. Ball 1985).
9.4 Incorporating variable utilization rates for capital

There is considerable evidence that utilization rates of capital vary significantly over time. This implies that firms generally are not in long-run equilibrium, and in turn that (a) flows of capital services are not in fixed proportion to the levels of capital stocks and (b) the marginal value product of capital services is not equal to a market wage or rental rate. Nevertheless the index number procedures of sections B-C and many econometric models incorporate these assumptions.

One approach to avoiding or reducing these problems in econometric models stems from the assumption that the rate of depreciation for capital varies with the utilization rate of capital. Then the firm’s short-run profit maximization problem can be written as

$$\max_{(x_t, K_t^f)} p^f(x_t, K_t, K_t^f) - w x_t + \tilde{p}^K K_t^f \equiv \pi(w, p, p^k, K_t)$$

(9.44)

where $K_t \equiv$ capital stock predetermined at beginning of period $t$, $K_t^f \equiv$ position of $K_t$ remaining at end of period $t$, and $\tilde{p}^K \equiv p^K/(1 + r)$ is the asset price of capital at the end of period $t$ discounted back to the beginning of period $t$. The derivatives of the production function are $f_x() > 0$, $f_K() > 0$ and $f_{t}(), < 0$ (an increase in $K_t^f$ for a given initial stock $K_t$ implies a lower depreciation rate and utilization rate of the initial stock $K_t$). However there is a serious difficulty in implementing models such as (9.44) where the depreciation rate as well as utilization rate of capital is treated as endogenous: time series data on capital stocks have invariably been constructed assuming a constant rate of depreciation, and there are substantial econometric problems in the estimation of (9.44) where $K_t$ is unobserved (see Epstein and Denny, 1980 for an illustration). Hereafter we shall assume for simplicity that variations in capital utilization do not influence the rate of depreciation.

Next consider the effects of short-run equilibrium on non-parametric Divisia index number calculations of technical change, as discussed in section B. Let the production function be $y = f(x, K, t)$ where $x$ denotes flows of variable inputs and $K$ denotes stocks of predetermined (quasi-fixed) capital inputs, and assume short-run competitive profit maximization

$$\max_x pf(x, K, t) - wx \equiv \pi(w, p, K, t)$$

(9.45)

Differentiating the identity $y(t) \equiv f(x(t), K(t), t)$ and applying the first order conditions $pf_x(x^*, K, t) - w^i = 0 \quad (i = 1, \cdots, N)$ for a solution to (9.45) yields

$$\frac{\partial f(x^*, K, t)}{\partial t} / y = \frac{\hat{y}}{y} - \sum_{i=1}^{N} \left( \frac{w^i x^i}{py} \right) \frac{\hat{x}^i}{x^i} - \sum_{j} \frac{\partial f(x^*, K, t)}{\partial K^j} K^j / y$$

(9.46)

However the marginal products of capital $\partial f(x^*, K, t) / \partial K^j$ are not observed directly are equal to price ratios $w^k/p$ only in a static long-run equilibrium.
Also note that the productivity index (9.46) employs the change in capital stocks $\dot{K}$ rather than a change in flow of capital services from the stocks. Equation (9.46) provides an exact measure of change in productivity in continuous time. Thus errors in calculations of changes in productivity $\frac{\partial f(x^*, K, t)}{\partial t}$ under the erroneous assumption of long-run equilibrium are due to mismeasurement of weights for changes in capital stocks $\dot{K}$ rather than to mismeasurement of the flow of capital services (the correct weights for $\dot{K}$ are the unobserved marginal products of capital). In other words, in productivity studies there is no need to derive capital service flows from capital stocks irrespective of the industry being in long-run or short-run equilibrium. This is an important result since the concept of capital service flows, which has been widely employed in earlier productivity studies, is an artificial construct with little empirical basis.

In the special case of constant returns to scale $f(\lambda x, \lambda K, t) = \lambda f(x, K, t)$ and one capital good $K$, the marginal product of capital can be calculated directly as

$$\frac{\partial f(x^*, K, t)}{\partial K} = \left[ y - \sum_{i=1}^{N} \left( \frac{w_i}{p} \right) x^i \right] / K$$  \hspace{1cm} (9.47)

Using Euler’s theorem and first order conditions for (9.45). Then changes in productivity can be calculated directly from a discrete approximation to (9.46).

Similarly the variable cost function $C = C(w, y, K, t) \equiv wx$ and short-run competitive behavior (9.45) imply

$$\frac{\partial C(w, y^*, K, t)}{\partial t} / C = \sum_{i=1}^{N} \left( \frac{w_i}{C} \right) \frac{\dot{K}_i}{x^i} - \sum_{j=1}^{M} \left( \frac{p^j y_i}{C} \right) \dot{y}_j / y_j \sum_{i} \frac{\partial C(w, y^*, K, t)}{\partial K^i} \dot{K}_i / C$$  \hspace{1cm} (9.48)

In the case of constant returns to scale in production and a single capital good $K$, the unobserved shadow price of capital can be calculated as

$$\frac{\partial C(w, y, K, t)}{\partial K} = \left( C - \sum_{j=1}^{M} p^j y_j \right) / K < 0$$  \hspace{1cm} (9.49)

(using $C(w, \lambda y, \lambda K, t) = \lambda C(w, y, K, t)$ and Euler’s theorem).

Substituting (9.49) into (9.48), the resulting index number formula for technical change can be approximated using data for discrete time intervals. This calculation of technical change under constant returns to scale and a single capital good $K$ requires neither econometrics nor the assumption of long-run equilibrium.

Next consider the effects of short-run equilibrium on parametric Divisia index number calculations of technical change, as discussed in section C. Assuming a Translog variable cost function $C(w, y, K, t)$ and competitive short-run profit maximization ((9.45)), equation (9.35) must be rewritten as
9.4 Incorporating variable utilization rates for capital

\[
\frac{1}{2} \left( T^C_1 + T^C_0 \right) = \left\{ \log \left( \frac{C_1}{C_0} \right) - \frac{1}{2} \sum_{i=1}^{N} \left[ \frac{w^i_1 x^i_1}{C_1} + \frac{w^i_0 x^i_0}{C_0} \right] \log \left( \frac{w^i_1}{w^i_0} \right) \right\}
\]

\[
-\frac{1}{2} \sum_{j=1}^{M} \left[ \frac{p^j_1 y^j_1}{C_1} + \frac{p^j_0 y^j_0}{C_0} \right] \log \left( \frac{y^j_1}{y^j_0} \right)
\]

\[
-\frac{1}{2} \sum_{j} \left[ \frac{\partial \log C_1}{\partial \log K_j} + \frac{\partial \log C_0}{\partial \log K_j} \right] \log \left( \frac{K^j_1}{K^j_0} \right)
\]

(9.50)

In general \( \frac{\partial \log C_j}{\partial \log K^j} \) cannot be measured without recourse to econometric estimation of the factor demand equations. However, in the case of a single quasi-fixed capital good \( K \) and constant returns to scale in production, \( \frac{\partial \log C_j}{\partial \log K^j} \) can be calculated from (9.49). Thus (as in non-parametric Divisia index number formulas) given the assumptions of constant returns to scale and one quasi-fixed input, parametric measures of technical change can be obtained without recourse to econometrics or the assumption of long-run equilibrium.

Thus, if there is more than one quasi-fixed input or returns to scale are not constant, econometric methods are required for the measurement of technical change. For example, we could postulate a short-run Translog cost function \( C(w, y, K, t) \) analogous to (9.1) and estimate factor share equations for the variable inputs. Then the change in technology \( \frac{\partial C(w, y, K, t)}{\partial t} \) can be calculated directly from the estimates of the share equations as discussed in section A (page 9.4). Alternatively, the shadow prices can be calculated as

\[
\frac{\partial C(w, y, K, t)}{\partial K^j} = \sum_{i=1}^{N} w^i \frac{\partial x^i(w, y, K, t)}{\partial K^j}
\]

(9.51)

From the estimates of the share equations, and then the change in technology \( \frac{1}{2} \left( T^C_1 + T^C_0 \right) \) can be calculated from the index number equation (9.50). The disadvantage of this second approach is that equation (9.50) requires the assumption of short-run competitive profit maximization, whereas estimation of the cost function only requires the weaker assumption of short-run competitive cost minimization.

We conclude this section with a brief discussion of empirical measures of capacity utilization based on microeconomic theory. Unexpected changes in output prices or input prices are likely to lead to short-run combinations of variable and quasi-fixed inputs that are inappropriate for the long-run, i.e., under or over-utilization of capacity is likely in the short-run. A fruitful approach to the measurement of capacity utilization is by comparing the observed level of output and “capacity output”. One definition of capacity output is the output level corresponding to the minimum point on the firm’s long-run average cost curve, but this definition is not useful due to difficulties in identifying the long-run average cost curve.

A more useful definition of capacity output is the output level \( y^C \) at which the short-run average total cost curve (with quasi-fixed inputs fixed at their short-run equilibrium levels) is tangent to the long-run average cost curve. If there is constant
returns to scale in the long-run, then capacity output corresponds to the minimum point on the short-run average total cost curve (see diagram on following page). The rate of capacity utilization $CU$ is then defined as the ratio of actual output $y$ to capacity output $y^C$:

$$CU = \frac{y}{y^C}$$  \hspace{1cm} (9.52)

Econometric estimates of a short-run cost function $C(w, y, K, t)$ can be employed in calculations of a capacity utilization index $CU$ (Berrdt and Herse 1986).

9.5 Conclusion

In this chapter we have discussed several non-econometric and econometric approaches to the measurement of technical change. The non-econometric approaches do not require a long time series of data or the use of highly aggregated data. On the other hand these approaches cannot test hypothesis about technical change, and these approaches can disentangle shifts in productivity and variations in capital utilization rates only in the case of constant returns to scale and a single quasi-fixed input. In contrast econometric approaches require a substantial number of observations and substantial aggregation of commodities, but these approaches can test hypothesis and can calculate changes in productivity given multiple quasi-fixed inputs.

In practice non-econometric and econometric approaches to the measurement of technical change can often be viewed as complementary. Most of the effort required to construct non-econometric.

Measures of (full) Capacity Output

A.

\[ SRAC \equiv AC(w, w^K, y, K) \]

\[ LRAC \equiv AC(w, w^K, y) \]

**Fig. 9.1** figure 1
B. Assuming constant returns to scale in the production function \( y = f(x, K) \), \( (f(\lambda x, \lambda K) = \lambda f(x, k)) \):

\[
SRAC_0 \equiv AC(w, w^K, y, K_0) \\
SRAC_1 \equiv AC(w, w^K, y, K_1)
\]

Fig. 9.2 figure 2

Short-run average cost function (SRAC)

\[
AC(w, w^K, y, K) = \min_{x} \frac{wx + w^K K}{y} \\
\text{s.t. } f(x, K) = y
\]

Long-run average cost function (LRAC):

\[
AC(w, w^K, y) = \min_{x, K} \frac{wx + w^K K}{y} \\
\text{s.t. } f(x, K) = y
\]

Index number calculations of technical change will also be required to construct econometric models, and data problems may be indicated more clearly by non-econometric measures of technical change. Thus the following sequence may be appropriate: first construct non-econometric index number measures of technical change, and then (data permitting) estimate a short-run equilibrium econometric model.

However at least two serious problems remain for the econometric models and Divisia index number formulas outlined here. First technical change is proxied by a simple time trend \( t = 1, 2, 3, \ldots \) (which may interact with other variables) or is calculated as a simple residual. Thus there is no theory endogenizing technical
change to the model. In principle a well constructed theory of demand and supply for changes in technology should improve both econometric and non-econometric measures of technical change. Second, there has been little attempt to incorporate explicitly into the analysis changes in quality of inputs supplied to the industry (see Berrdt 1983 and Tarr 1982).

4. It should be noted that cross-sectional differences in productivity (e.g. differences in productivity between countries, regions or firms) can be measured by parametric index number procedures somewhat similar to the procedures discussed in section C (Caves, Christensen and Diewert 1982; Denny and Fuss 1983).
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