Department of Electrical and Computer Engineering

University of Manitoba

Ph.D. Candidacy Examination in Telecommunications

Winter 2019

READ THE FOLLOWING INSTRUCTIONS CAREFULLY

- The duration of the examination will be 3 hours.
- A non-programmable calculator is allowed. No other electronic devices are permitted.
- The examination will be **closed-book**. No printed material is allowed. However, the candidates are allowed to bring a **single** letter size, **hand-written** formula sheets (photocopies are not permitted).
- Answer 4 problems. All 4 carry equal marks.
- Start each problem on a new page.
- Hand-in all three of the following at the conclusion of the exam.
  - your answer booklet
  - this exam paper
- **Important**: The goal of this examination is to evaluate your fundamental understanding of the broad area of communication theory and systems. It is your responsibility to provide as much explanation as required to demonstrate your thought process. In particular, you must clearly describe how you arrive at each answer. Provide and justify all important intermediate steps in analytical work. Intermediate steps and the accompanying reasoning carry most of the marks. Answers with no steps or justifications receive zero marks.

1. **Probability theory and Bayesian inference**

The data $x[n] = A + w[n]$ (for $n = 0, 1, ..., N-1$) are observed in which $\{w[n]\}_{n=0}^{N-1}$ are samples of a zero-mean white Gaussian noise with variance σ_w^2. The goal is to estimate the unknown parameter A from the N observations $\{x[n]\}_{n=0}^{N-1}$.

1) Find the maximum likelihood estimate of A (**5 points**).
2) Find the maximum *a posteriori* (MAP) estimate of A under the following three different priors, $p_A(A)$, on A. In all three cases, we assume that A is independent of $\{w[n]\}_{n=0}^{N-1}$:

   (a) **Uniform prior:** $A_0 > 0$ is known (**5 points**):

$$p_A(A) = \begin{cases} \frac{1}{2A_0} & |A| \leq A_0 \\ 0 & |A| > A_0 \end{cases} \tag{1}$$

   (b) **Exponential prior:** $\lambda > 0$ is known (**5 points**):

$$p_A(A) = \begin{cases} \lambda \exp(-\lambda A) & A > 0 \\ 0 & A \leq 0 \end{cases} \tag{2}$$

   (c) **Gaussian prior:** $\sigma_A^2 > 0$ and μ_A are known (**10 points**):

$$p_A(A) = \frac{1}{\sqrt{2\pi\sigma_A^2}} \exp\left[-\frac{(A-\mu_A)^2}{2\sigma_A^2}\right], \tag{3}$$

2. **Stochastic Processes**

(a) Give precise definitions of

i. strictly stationary random process

ii. weakly stationary random process

iii. Gaussian random process

(b) Prove that if $X(t)$ is a weakly-stationary Gaussian random process, then it is also strictly stationary, where t is continuous time.

(c) Consider the continuous-time random process

$$Y(t) = A\cos^2(2\pi f_0 t + \Theta),$$

where A and Θ are independent random variables and f_0 is a constant. The probability density functions of A and Θ respectively are

$$f_A(a) = \begin{cases} 2e^{-2a}, & a \geq 0 \\ 0, & a < 0 \end{cases}$$

and

$$f_\Theta(\theta) = \begin{cases} \text{a constant}, & |\theta| \leq 2\pi \\ 0, & \text{otherwise.} \end{cases}$$

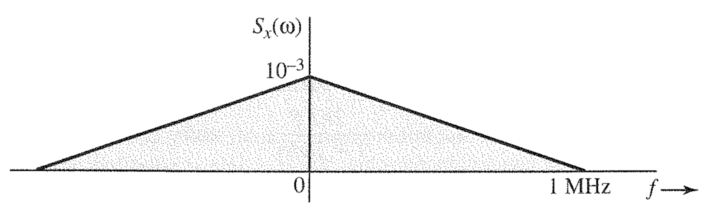
i. Find the mean-value of $Y(t)$ as a function of t.

ii. Find the auto-correlation function of $Y(t)$.

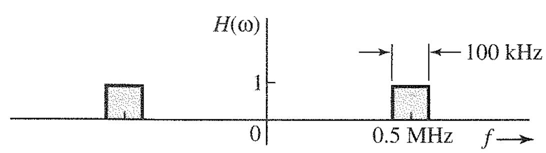
iii. Is $Y(t)$ weakly stationary? Justify.

iv. Is $Y(t)$ an ergodic process? Justify.

(d) A Gaussian random process $X(t)$ with the power spectral density function $S_X(f)$ is passed through an ideal bandpass filter whose frequency response is $H(f)$. $S_X(f)$ and $H(f)$ are as shown below.



(a)



(b)

i. Determine the power spectral density functions of the inphase and quadrature components of the filter output.

ii. Determine the joint probability density function of the inphase and quadrature components of the filter output.

3. **Information Theory**

(a) An analog signal $x(t)$ with bandwidth of 6 kHz is sampled at the Nyquist rate. Assume that the consecutive samples are statistically independent. Each sample $x(n)$ is digitized using a quantizer whose output levels and their probability of occurrences are as follows.

Output level	-0.4	-0.1	0.1	0.4
Probability	0.15	0.35	0.35	0.15

The output of the quantizer is to be compressed with a lossless code. Determine the *minimum* possible bit-rate (in bits/sample) of the lossless encoder output if the quantizer output is encoded

i. in a sample-by-sample manner using a binary code

ii. in a sample-by-sample manner using a code with the alphabet $\{0, 1, 2, 3\}$

iii. using a Lempel-Ziv code (find a good estimate of what is possible, stating any assumptions made)

(b) Consider a discrete communication channel whose input $X \in \{-1, 1\}$ and output $Y \in \{-1, 0, 1\}$. Regardless of the value of X, the conditional probability that $Y = X$ is μ and the conditional probability that $Y = 0$ is $1 - \mu$. A sequence of binary source symbols $S_1, S_2, \ldots$ with probabilities $P(S_n = 0) = p$ and $P(S_n = 1) = 1 - p$, $n = 1, 2, \ldots$, is to be transmitted error-free over this channel. Determine the maximum value of p for which we could transmit at the rate one source symbol per channel symbol, i.e., we transmit a source codeword $S_1, S_2, \ldots, S_L$ using a channel codeword $X_1, X_2, \ldots, X_L$ with L appropriately chosen.

(c) A bandlimited analog signal is to be transmitted over an additive white Gaussian noise (AWGN) channel whose output signal-to-noise ratio (SNR) is 12 dB. For this purpose, the analog signal is first sampled at the rate of 8000 samples/sec. Assume that the sampled signal x_n (n is discrete-time) can be well modeled as an uncorrelated zero mean Gaussian random process with the variance 0.25. The sequence of samples are then optimally encoded (using source and channel coding) and transmitted over the AWGN channel. Determine the minimum channel bandwidth required, if we wish to achieve a mean square error of no larger than 0.01 in reconstructing the sampled signal x_n at the receiver. Clearly show how you arrive at your answer.

4. **Signaling over AWGN channels**

Preliminaries: The correlation coefficient, ρ, of any two signals $f(t)$ and $g(t)$ is defined as follows:

$$\rho \triangleq \frac{1}{\sqrt{\mathcal{E}_f \mathcal{E}_g}} \int_{-\infty}^{+\infty} f(t)g(t)dt \tag{4}$$

where $\mathcal{E}_f$ and $\mathcal{E}_g$ are the energies of the signals $f(t)$ and $g(t)$, respectively.

The normalized sinc function is also defined for all $t \in \mathbb{R}^*$ as follows:

$$\text{sinc}(t) = \begin{cases} \frac{\sin(\pi t)}{\pi t}, & t \neq 0 \\ 1, & t = 0 \end{cases} \tag{5}$$

The sinc function is plotted in Fig. 1 for $-5 \leq t \leq 5$.

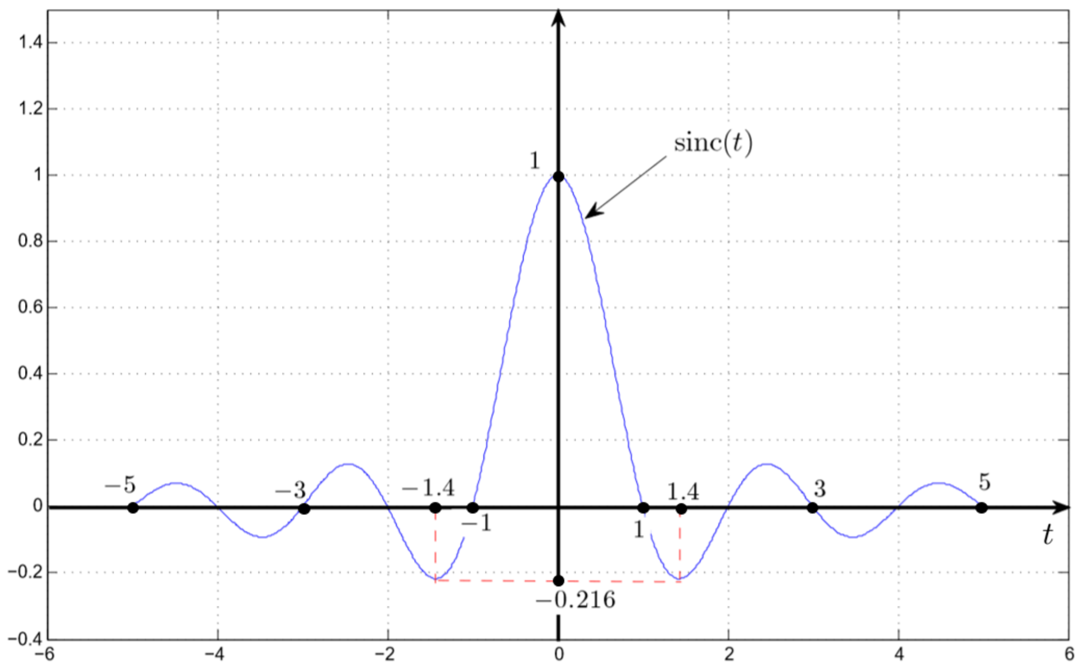


Figure 1: Plot of the sinc function defined in eq. (5).

Problem Statement: A binary frequency-shift keying (FSK) system, transmits either $s_1(t)$ or $s_2(t)$ to convey the information bits 0 or 1, respectively:

$$s_1(t) = A\cos\left(2\pi\left[f_c + \frac{\Delta f}{2}\right]t\right) \qquad 0 \leq t \leq T, \tag{6}$$

$$s_2(t) = A\cos\left(2\pi\left[f_c - \frac{\Delta f}{2}\right]t\right) \qquad 0 \leq t \leq T, \tag{7}$$

in which $A = \sqrt{2\mathcal{E}_b/T}$, $f_c \gg 1$, and $\Delta f > 0$ with $\Delta f \ll f_c$. We also assume that the two signals are equally likely. The received signal is $r(t) = s_m(t) + n(t)$, $m = 1, 2$, with the additive noise, $n(t)$, being well-modeled by a zero-mean Gaussian process with power spectral density $S_N(f) = \frac{N_0}{2}$, $\forall f$.

1.1 a) Show that the correlation coefficient, ρ, of the two signals $s_1(t)$ or $s_2(t)$ is approximately given by (**6 points**)

$$\rho \approx \text{sinc}\,(2\Delta f T) \tag{8}$$

b) Use Fig. 1 to sketch the plot of the approximate expression of ρ in (8) as function of Δf. In your plot, you must show the values of the key points (along the $x-$ and $y-$ axes) as already done in Fig. 1 for the pure sinc function. (**2 points**).

1.2 What is the minimum value of frequency shift Δf for which the two signals $s_1(t)$ or $s_2(t)$ are approximately orthogonal? (**1 point**).

1.3 Express the average probability of error, P_e, of the underlying binary FSK system as function of ρ (**10 points**).
Hint: You should first express P_e as function of the distance, d_{12}, between the two signals $s_1(t)$ and $s_2(t)$.

1.4 Deduce the value (as function of T) of Δf that minimizes P_e. (**2 points**).

1.5 For the value of Δf found in 1.4, determine the increase in the SNR per bit, $\mathcal{E}_b/N_0$, required so that this binary FSK system has the same average probability of error as a binary antipodal system (**4 points**).