PhD Candidacy Exam - Signal Processing

1. Given a continuous-tome signal $x(t)$ with $X^F(\omega) = 0$ for $|\omega| > \omega_m$ determine the minimum sampling rate f_s for a signal $y(t)$ defined by

   (a) $x^2(t)$

   (b) $x(2t)$

   (c) $x(t)\cos(6\pi\omega_m t)$

2. (a) Find the period of the signal $f[k] = e^{j\frac{\pi}{16}k}\cos\left(\frac{\pi}{17}k\right)$.

   (b) Find the energy and power of the signal $f[k] = (1.5)^k\, u[-k]$.

   (c) For a DT system with input $f[k]$ and output $y[k]$, if $y[k] = f[k]\sin(\frac{\pi}{2}k)$, determine whether the system is *linear time invariant (LTI)* or not.

   (d) Determine whether the system defined by $y[k] = f[k] * \cos(\frac{\pi}{8}k)$ BIBO stable or not.

3. (a) A system's response to $x(n)$ is $y(n)$ as the followings:

$$x(n) = (\frac{1}{2})^n u(n) - \frac{1}{4}(\frac{1}{2})^{n-1}u(n-1)$$

$$y(n) = (1/3)^n u(n)$$

   i. Determine the impulse response $h(n)$ and the system function $H(z)$ of a system that satisfies the above condition.

   ii. Find the difference equation that characterizes this system.

   iii. Determine if the system is stable.

   (b) Using the properties of Z-Transform, show that for a real $x(n)$ that has a $H(z)$, if a pole (zero) of $H(z)$ occurs at $z = z_0$, then a pole (zero) must also occur at $z = z_0^*$.

4. Consider signal $x[n] = \{-1, 2, -3, 2, -1\}$ with discrete-time Fourier transform $X^f(\theta)$. Compute the following quantities:

   (a) $X^f(0)$

   (b) $\int_{-\infty}^{\infty} X^f(\theta)d\theta$

   (c) $X^f(\pi)$

   (d) $\int_{-\pi}^{\pi} |X^f(\theta)|^2 d\theta$

5. (a) Let $x(n)$, $n = 0, 1, \ldots, 7$ be a purely imaginary sequence. The first five points of an eight-point DFT of this sequence are

$$\{j0.55, -0.055 + j0.1636, 0.3 + j0.15, 1.0450 + j1.4364, -j0.05\}.$$

Determine the remaining three points and explain the basis of your answer.

(b) Let $y(n)$, $n = 0, \ldots, N-1$ be a complex sequence, whose N-point DFT is $Y(k) = \alpha_k$, $k = 0, \ldots, N-1$. Find an expression for the N-point DFT of the complex conjugate of $y(n)$, in terms of $\alpha_0, \ldots, \alpha_{N-1}$.

(c) We wish to obtain the spectrum of an analog signal whose bandwidth is 8 kHz, by first sampling the signal and then computing the DFT. For practical reasons, we have to use the minimum possible sampling frequency. If we desire a spectral resolution of at least 40 Hz, determine the minimum allowable duration of the analog signal record. Clearly explain how you arrive at your result.

(d) A 7.5 kHz sinusoidal signal is sampled at the rate of 20 kHz, and 256 samples are used to compute the 512-point DFT $X(k)$, $k = 0, \ldots, 511$. At which k would you expect to see a peak in $X(k)$? Explain the reasoning behind your answer.

6. Consider the single-pole digital filter given by

$$y(n) = Q_b[ay(n-1)] + x(n),$$

where $x(n)$ and $y(n)$ are the input and output respectively, a is the filter coefficient and $Q_b[\cdot]$ denotes the b-bit round-off operation in the digital-multiplier.

(a) One simple way to analyze the effect of round-off noise is to represent the round-off operation by the *additive noise model* for quantization errors. State the key assumptions made in this model.

(b) Using this model, derive a closed-form expression (in terms of a and b) for the variance of the error that will appear at the *output* of the filter. Clearly show how you arrive at the solution.

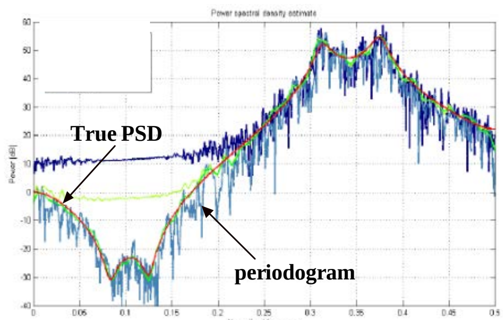
7. An AR(3) process is characterized by prediction coefficients: $a_3(1) = -1.25$, $a_3(2) = 1.25$, $a_3(3) = -1$

(a) Determine its lattice filter reflection coefficients.

(b) Determine its autocorrelation $r_{xx}(m)$ for $0 \le m \le 3$.

(c) Determine the mean-square prediction error.

8. Let $\{X(n)\}$ be a wide-sense real-valued stationary stochastic process with $E\{X(n)\} = 0$. Let $r(k) = E\{X(n)X(n-k)\}$, $k = 0, \pm 1, \pm 2, \ldots$ denote the autocorrelation function and $S(f) = \sum_{k=-\infty}^{\infty} r(k)e^{-j2\pi fk}$ the power spectral density (PSD) of the process.
Let $X(0), X(1), \ldots, X(N-1)$ be an observed data record of the process $\{X(n)\}$. We wish to estimate the PSD function $S(f)$ of $\{X(n)\}$ from the data record. The plot below shows a basic estimate of $S(f)$ referred to as the periodogram and denoted as $I_N(f)$. Also the true $S(f)$ is depicted.



(a) Explain why $I_N(f)$ reveals such high oscillations. Quantify this in terms of the bias and variance of $I_N(f)$. Is $I_N(f)$ converging to $S(f)$ as $N \to \infty$?

(b) Introduce the concept of windowed periodogram estimates and explain why this class of estimates can converge to the true $S(f)$. In particular, elaborate on the bias-variance trade-off characterizing this class of estimates. Give an example, e.g., Bartlett window, of the proper window function yielding a consistent estimate of $S(f)$.

(c) The following plot shows various estimates $\hat{S}_N(f)$ of $S(f)$ based on $N = 1024$ data points. It is assumed that the data were generated from the following sinusoidal signal model

$$X(n) = A_1 e^{j(2\pi f_1 n + \theta_1)} + A_2 e^{j(2\pi f_2 n + \theta_2)} + w(n),$$

where $A_1, A_2, \theta_1, \theta_2$ are *iid* random variables uniformly distributed on $[0, 2\pi]$ and $w(n)$ is a zero-mean, white noise process. Based on this plot can you estimate the values of the unknown frequencies f_1 and f_2 ?