1. Given a continuous-time signal $x(t)$ with $X^F(\omega) = 0$ for $|\omega| > \omega_m$ determine the minimum sampling rate $f_s$ for a signal $y(t)$ defined by

(a) $x^2(t)$
(b) $x(2t)$
(c) $x(t) \cos(6\pi\omega_m t)$

2. (a) Find the period of the signal $f[k] = e^{j\frac{\pi}{17}k} \cos\left(\frac{\pi}{2}k\right)$.
(b) Find the energy and power of the signal $f[k] = (1.5)^k u[-k]$.
(c) For a DT system with input $f[k]$ and output $y[k]$, if $y[k] = f[k] \sin\left(\frac{\pi}{2}k\right)$, determine whether the system is linear time invariant (LTI) or not.
(d) Determine whether the system defined by $y[k] = f[k] \ast \cos\left(\frac{\pi}{8}k\right)$ BIBO stable or not.

3. (a) A system’s response to $x(n)$ is $y(n)$ as the followings:

$$x(n) = \left(\frac{1}{2}\right)^n u(n) - \left(\frac{1}{4}\right)^{n-1} u(n-1)$$

$$y(n) = (1/3)^n u(n)$$

i. Determine the impulse response $h(n)$ and the system function $H(z)$ of a system that satisfies the above condition.

ii. Find the difference equation that characterizes this system.

iii. Determine if the system is stable.

(b) Using the properties of $Z$-Transform, show that for a real $x(n)$ that has a $H(z)$, if a pole (zero) of $H(z)$ occurs at $z = z_0$, then a pole (zero) must also occur at $z = z_0^*$.

4. Consider signal $x[n] = \{-1, 2, -3, 2, -1\}$ with discrete-time Fourier transform $X^f(\theta)$. Compute the following quantities:

(a) $X^f(0)$
(b) $\int_{-\infty}^{\infty} X^f(\theta) d\theta$
(c) $X^f(\pi)$
(d) $\int_{-\pi}^{\pi} |X^f(\theta)|^2 d\theta$
5. (a) Let $x(n)$, $n = 0, 1, \ldots, 7$ be a purely imaginary sequence. The first five points of an eight-point DFT of this sequence are 

$$\{j0.55, -0.055 + j0.1636, 0.3 + j0.15, 1.0450 + j1.4364, -j0.05\}.$$ 

Determine the remaining three points and explain the basis of your answer.

(b) Let $y(n)$, $n = 0, \ldots, N - 1$ be a complex sequence, whose $N$-point DFT is $Y(k) = \alpha_k$, $k = 0, \ldots, N - 1$. Find an expression for the $N$-point DFT of the complex conjugate of $y(n)$, in terms of $\alpha_0, \ldots, \alpha_{N-1}$.

(c) We wish to obtain the spectrum of an analog signal whose bandwidth is 8 kHz, by first sampling the signal and then computing the DFT. For practical reasons, we have to use the minimum possible sampling frequency. If we desire a spectral resolution of at least 40 Hz, determine the minimum allowable duration of the analog signal record. Clearly explain how you arrive at your result.

(d) A 7.5 kHz sinusoidal signal is sampled at the rate of 20 kHz, and 256 samples are used to compute the 512-point DFT $X(k)$, $k = 0, \ldots, 511$. At which $k$ would you expect to see a peak in $X(k)$? Explain the reasoning behind your answer.

6. Consider the single-pole digital filter given by

$$y(n) = Q_b[ay(n - 1)] + x(n),$$

where $x(n)$ and $y(n)$ are the input and output respectively, $a$ is the filter coefficient and $Q_b[\cdot]$ denotes the $b$-bit round-off operation in the digital-multiplier.

(a) One simple way to analyze the effect of round-off noise is to represent the round-off operation by the additive noise model for quantization errors. State the key assumptions made in this model.

(b) Using this model, derive a closed-form expression (in terms of $a$ and $b$) for the variance of the error that will appear at the output of the filter. Clearly show how you arrive at the solution.

7. An AR(3) process is characterized by prediction coefficients: $a_3(1) = -1.25$, $a_3(2) = 1.25$, $a_3(3) = -1$

(a) Determine its lattice filter reflection coefficients.

(b) Determine its autocorrelation $r_{xx}(m)$ for $0 \leq m \leq 3$.

(c) Determine the mean-square prediction error.
8. Let \( \{X(n)\} \) be a wide-sense real-valued stationary stochastic process with \( E\{X(n)\} = 0 \). Let \( r(k) = E\{X(n)X(n-k)\}, k = 0, \pm 1, \pm 2, \ldots \) denote the autocorrelation function and \( S(f) = \sum_{k=-\infty}^{\infty} r(k)e^{-j2\pi fk} \) the power spectral density (PSD) of the process. Let \( X(0), X(1), \ldots, X(N-1) \) be an observed data record of the process \( \{X(n)\} \). We wish to estimate the PSD function \( S(f) \) of \( \{X(n)\} \) from the data record. The plot below shows a basic estimate of \( S(f) \) referred to as the periodogram and denoted as \( I_N(f) \). Also the true \( S(f) \) is depicted.

(a) Explain why \( I_N(f) \) reveals such high oscillations. Quantify this in terms of the bias and variance of \( I_N(f) \). Is \( I_N(f) \) converging to \( S(f) \) as \( N \to \infty \)?

(b) Introduce the concept of windowed periodogram estimates and explain why this class of estimates can converge to the true \( S(f) \). In particular, elaborate on the bias-variance trade-off characterizing this class of estimates. Give an example, e.g., Bartlett window, of the proper window function yielding a consistent estimate of \( S(f) \).

(c) The following plot shows various estimates \( \hat{S}_N(f) \) of \( S(f) \) based on \( N = 1024 \) data points. It is assumed that the data were generated from the following sinusoidal signal model

\[
X(n) = A_1e^{j(2\pi f_1 n + \theta_1)} + A_2e^{j(2\pi f_2 n + \theta_2)} + w(n),
\]

where \( A_1, A_2, \theta_1, \theta_2 \) are iid random variables uniformly distributed on \([0, 2\pi]\) and \( w(n) \) is a zero-mean, white noise process. Based on this plot can you estimate the values of the unknown frequencies \( f_1 \) and \( f_2 \)?