STUDENT NUMBER

STUDENT'S SIGNATURE ON THIS LINE

PRINT NAME IN FULL ON THIS LINE

**General Instructions:**

1) This is an **open-book** exam in the following sense: *you are allowed to bring in any one mathematics textbook of your choosing.*
2) There are a total of ten possible problems, Q1-Q10, in this exam. You are to **choose only five problems for marking**. On the list below circle the questions of the problems you want marked and only those will be marked. If none are circled the first five will be marked.
3) **No large memory programmable calculators are permitted.**
4) Cell phones and other wireless devices must be turned off.
5) You will be provided with scrap paper.
6) Make sure that your name, student number, and signature are written on this page.

| | |
|---|---|
| Q1: | /10 |
| Q2: | /10 |
| Q3: | /10 |
| Q4: | /10 |
| Q5: | /10 |
| Q6: | /10 |
| Q7: | /10 |
| Q8: | /10 |
| Q9: | /10 |
| Q10: | /10 |
| TOTAL: | **/ 50** |

Q1. Solve for $x_1$ and $x_2$ in the following system of equations:

$$\begin{cases} 3.5x_1 + 0.45\tanh(x_1 + x_2) = 4.47 \\ 2.75x_1 + 0.9\tanh(x_1 + x_2) = -0.83 \end{cases}$$

Q2. Perform an LU decomposition of

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

and use it to solve for $Ax = b$ where $b = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T$. Is $A$ positive definite?

Q3. Consider the following optimization problem:

$$\underset{x}{\text{minimize}} \quad f(x) = (x_1 - 1)^2 + (x_2 - 1)^2, \; x \in \mathbb{R}^2$$

$$\text{subject to:} \quad \begin{aligned} g_1(x) &= x_1 + x_2 - 1 \geq 0 \\ g_2(x) &= 1 - x_1 \geq 0 \\ g_3(x) &= 1 - x_2 \geq 0 \end{aligned}$$

Solve it using the following penalty function form: $P(x, R) = f(x) + R \sum_{i=1}^{3} g_i^{-1}(x)$.

Find the stationary points of $P$ as a function of $R$ and let $R$ vary appropriately. Show the results in a table and comment.

Q4. Consider the following positive-definite quadratic objective function minimization problem:

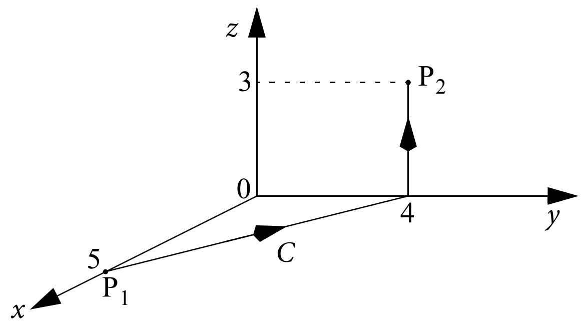
$$\underset{x}{\text{minimize}} \quad f(x) = \|Ax + b\| \text{ subject to: } Cx = d$$

where $x, b \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$, $d \in \mathbb{R}^m$ and $C \in \mathbb{R}^{m \times n}$ with $m < n$ and $\text{rank}(C) = m$. Determine a closed-form expression for the solution to the problem.

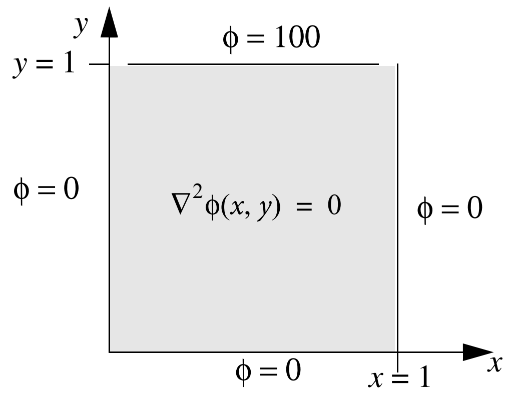
Q5. Find the gradient of $\Phi(x, y, z)$, $\boldsymbol{E}(x, y, z) = -\nabla\Phi(x, y, z)$ where,

$$\Phi(x, y, z) = e^{\alpha x}\cos(\beta y)\cosh(\gamma z).$$

Now find the line integral $\int_{P_1}^{P_2} \boldsymbol{E} \cdot \boldsymbol{dl}$ from point $P_1$ to point $P_2$ along the curve $C$ shown in the figure.



Q6. Find a solution for Laplace's equation in 2D with the boundary conditions shown.



Q7. Consider the first-order system of ordinary differential equations

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t),$$
$$\boldsymbol{x}(0) = \boldsymbol{x}_0$$

where $\boldsymbol{x}(t) \in \mathbb{R}^n$ is the solution vector, $\boldsymbol{u}(t) \in \mathbb{R}^r$ is the excitation vector, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times r}$ are constant coefficient matrices, and $\boldsymbol{x}_0 \in \mathbb{R}^n$ is the vector of initial conditions. Describe how you would solve this system (*i.e.*, write a formal solution in terms of $A$, $B$, $\boldsymbol{x}_0$, and $\boldsymbol{u}(t)$.

Q8. Use the Laplace Transform technique to solve the following first-order system of ordinary differential equations

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t),$$

with

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{u}(t) = \begin{cases} 0 & t < 0 \\ \cos\omega_0 t & t \ge 0 \end{cases} .$$

Is the system stable for all values of $\omega_0$?

Q9. Find the Fourier series on $-\pi < x < \pi$ of $|\sin x|$.

Q10. We'd like to model a signal, $s(t)$, as a damped exponential using the formula

$$s(t) = a(10^{bt})$$

The coefficient $a$ and the exponent $b$ are to be determined such that the sum of the squares of the errors between the model and the actual data is minimized.

a) Given the following experimental data. Write this problem as an over-determined system in the form $A\boldsymbol{c} = \boldsymbol{d}$ where the elements of the vector $\boldsymbol{c}$ are related to the unknown parameters $a$ and $b$.

| n | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| time, $t_n$ | 0 | 0.21 | 0.39 | 0.61 |
| signal, $s(t_n)$ | 1000 | 100 | 10 | 1 |

b) Without actually solving for the numerical values, show how you would go about solving for the parameters $a$ and $b$.