Optimal Asset Allocation for Endowment Management

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Outline

1. Objective
2. Theoretical Background
3. Statement of Problem
4. Numerical Analysis and Results
5. Long-Term Fund Evolution
6. Conclusions
Objective

- Stable Spending Rate and Long-Term Sustainability
- Capital Preservation

Research Objective:
Develop an Asset Allocation Framework and Determine Optimal Portfolio Selection for

- Sustainable Spending Rate (SSR) portfolio - a portfolio that allows sustainability of the highest spending rate that still keeps the endowment fund from default.
- Safety First Portfolio (SF) portfolio - a portfolio that results in the lowest probability of ruin for a desired spending rate.
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The inverse of the present value of a stochastic perpetuity based on Weiner returns is Gamma distributed.

**Theorem (Milevsky 1997)**

Let the present value of the stochastic perpetuity of 1, be

\[ X = \int_{0}^{\infty} e^{-t \cdot tr_t} dt \]

where \( tr_t \), the rate of return on \( (0, t) \), is a Wiener process \( \mu t + \sigma B_t \).

Then \( X^{-1} \sim \text{Gamma}(x; \frac{2\mu}{\sigma^2}, \frac{\sigma^2}{2}) \).
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Then \( X^{-1} \sim \text{Gamma}(x; \frac{2\mu}{\sigma^2}, \frac{\sigma^2}{2}) \).

Consider a portfolio with $n$ assets. $\mathbf{x} \in \mathbb{R}^n$ random vector of asset returns.

- $\mathbf{w}$ is the $n \times 1$ vector of asset weights in the portfolio s.t. $\mathbf{w}^T \mathbf{1} = 1$

- Assume that for $i = 1, \ldots, n$, $x_i \sim N(\mu_i, \sigma_i^2)$, where
  - $\mu_i = E[x_i]$, the mean of the $i$-th asset’s return
  - $\sigma_i^2 = \text{Var}(x_i)$, the variance of the $i$-th asset’s return

Therefore the return on the entire portfolio, $x_p = \sum_{i=1}^n w_i x_i \sim N(\mu_p, \sigma_p^2)$

- $\mu_p = \sum_{i=1}^n w_i \mu_i$
- $\sigma_p^2 = \sum_{i=1}^n \sum_{j=1}^n [w_i \Sigma_{ij} w_j]$, where $\Sigma = [\sigma_{ij}]$ is the covariance matrix with entries $\sigma_{ij} = \text{Cov}(x_i, x_j)$.

The mean-variance principle (Markowitz (1952)) enables to identify portfolios $(\mathbf{w})$ that

- have maximal return for given risk
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Problem 1 - SSR Portfolio

For a given probability of ruin, $\bar{\pi}$, find the optimal portfolio (optimal $w$) and the associated highest sustainable spending rate.

i.e., find

$$\max_w s(w) = \text{Gamma}^{-1}(\bar{\pi}; 2\mu_p/\sigma_p^2, \sigma_p^2/2)$$

subject to

$$\sum_{i=1}^{n} w_i = 1$$

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## Data Set

### Data: Jan 1990 to Dec 2010

<table>
<thead>
<tr>
<th>Asset Class</th>
<th>Mean</th>
<th>Standard Dev</th>
<th>Median</th>
<th>Sharpe</th>
</tr>
</thead>
<tbody>
<tr>
<td>U.S. Equity</td>
<td>2.84</td>
<td>15.32</td>
<td>0.09</td>
<td>0.2</td>
</tr>
<tr>
<td>Non U.S. Equity</td>
<td>-0.91</td>
<td>18.02</td>
<td>0.07</td>
<td>-0.04</td>
</tr>
<tr>
<td>U.S. Fixed Income</td>
<td>-3.28</td>
<td>3.76</td>
<td>-0.01</td>
<td>-0.83</td>
</tr>
<tr>
<td>Non U.S. Fixed Income</td>
<td>3.58</td>
<td>9.05</td>
<td>0.02</td>
<td>0.41</td>
</tr>
<tr>
<td>Private Real Estate</td>
<td>3.04</td>
<td>3.01</td>
<td>0.05</td>
<td>1.06</td>
</tr>
<tr>
<td>Hedge Funds</td>
<td>7.98</td>
<td>7.06</td>
<td>0.11</td>
<td>1.16</td>
</tr>
<tr>
<td>Venture Capital</td>
<td>10.96</td>
<td>12.47</td>
<td>0.07</td>
<td>0.89</td>
</tr>
<tr>
<td>Private Equity</td>
<td>9.96</td>
<td>6.02</td>
<td>0.12</td>
<td>1.68</td>
</tr>
<tr>
<td>Natural Resources</td>
<td>1.04</td>
<td>22.02</td>
<td>0.04</td>
<td>0.005</td>
</tr>
<tr>
<td>Cash</td>
<td>-0.16</td>
<td>0.54</td>
<td>0.00</td>
<td>NA</td>
</tr>
<tr>
<td>Inflation*</td>
<td>3.57</td>
<td>0.37</td>
<td>3.46</td>
<td>NA</td>
</tr>
</tbody>
</table>

* Inflation data follows HEPI (Higher Education Price Index)

In order to compute $\mu_p$ and $\sigma_p$ for the portfolio invested in the 10 assets, the vector of mean returns and the covariance matrix of returns are needed to be determined.

- Find the $10 \times 10$ covariance matrix $\hat{\Sigma}$ using various estimators (e.g., MLE, Hubert, Hellinger)

- Find the $10 \times 1$ vector of mean returns $\hat{\mu}$ using Reverse Optimization CAPM (Capital Asset Pricing Model)

$$\hat{\mu} = A\hat{\Sigma}w$$

where the risk aversion coefficient $A = 4$ and the asset weights used are the average market portfolio weights.
Parameter Estimation

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The resulting $\mu$ and $\sigma$ using MLE:

<table>
<thead>
<tr>
<th>Asset Class</th>
<th>$\mu$</th>
<th>$\sigma$</th>
<th>$w$</th>
</tr>
</thead>
<tbody>
<tr>
<td>U.S. Equity</td>
<td>3.94</td>
<td>15.32</td>
<td>15.00</td>
</tr>
<tr>
<td>Non U.S. Equity</td>
<td>4.79</td>
<td>18.02</td>
<td>16.00</td>
</tr>
<tr>
<td>U.S. Fixed Income</td>
<td>0.17</td>
<td>3.76</td>
<td>10.80</td>
</tr>
<tr>
<td>Non U.S. Fixed Income</td>
<td>0.56</td>
<td>9.05</td>
<td>1.20</td>
</tr>
<tr>
<td>Private Real Estate</td>
<td>0.19</td>
<td>3.01</td>
<td>5.20</td>
</tr>
<tr>
<td>Hedge Funds</td>
<td>1.81</td>
<td>7.06</td>
<td>23.92</td>
</tr>
<tr>
<td>Venture Capital</td>
<td>1.63</td>
<td>12.47</td>
<td>3.12</td>
</tr>
<tr>
<td>Private Equity</td>
<td>1.06</td>
<td>6.02</td>
<td>12.48</td>
</tr>
<tr>
<td>Natural Resources</td>
<td>3.15</td>
<td>22.02</td>
<td>7.28</td>
</tr>
<tr>
<td>Cash</td>
<td>0.03</td>
<td>0.54</td>
<td>5.00</td>
</tr>
</tbody>
</table>

* source: NACUBO-Commonfund Study of Endowments 2010
Results shown for MLE

<table>
<thead>
<tr>
<th>Probability of Ruin, $\pi$ (input)</th>
<th>1%</th>
<th>5%</th>
<th>10%</th>
</tr>
</thead>
<tbody>
<tr>
<td>U.S. Equity</td>
<td>31.17</td>
<td>36.69</td>
<td>34.49</td>
</tr>
<tr>
<td>Non U.S. Equity</td>
<td>31.19</td>
<td>45.54</td>
<td>53.68</td>
</tr>
<tr>
<td>U.S. Fixed Income</td>
<td>0.06</td>
<td>0.01</td>
<td>0.00</td>
</tr>
<tr>
<td>Non U.S. Fixed Income</td>
<td>0.05</td>
<td>0.01</td>
<td>0.00</td>
</tr>
<tr>
<td>Private Real Estate</td>
<td>0.05</td>
<td>0.01</td>
<td>0.00</td>
</tr>
<tr>
<td>Hedge Funds</td>
<td>14.72</td>
<td>0.02</td>
<td>0.01</td>
</tr>
<tr>
<td>Venture Capital</td>
<td>9.35</td>
<td>0.10</td>
<td>0.01</td>
</tr>
<tr>
<td>Private Equity</td>
<td>0.27</td>
<td>0.01</td>
<td>0.01</td>
</tr>
<tr>
<td>Natural Resources</td>
<td>13.09</td>
<td>14.60</td>
<td>11.78</td>
</tr>
<tr>
<td>Cash</td>
<td>0.04</td>
<td>0.01</td>
<td>0.00</td>
</tr>
<tr>
<td>Optimal Spending Rate ($s$)</td>
<td>0.90</td>
<td>1.45</td>
<td>1.85</td>
</tr>
<tr>
<td>Expected Return ($\mu_p$)</td>
<td>3.56</td>
<td>4.21</td>
<td>4.31</td>
</tr>
<tr>
<td>Risk ($\sigma_p$)</td>
<td>11.98</td>
<td>14.43</td>
<td>14.85</td>
</tr>
</tbody>
</table>
## SF Portfolio: Numerical Results

Results shown for MLE

<table>
<thead>
<tr>
<th>Spending rate $s$ (input)</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>U.S. Equity</td>
<td>32.21</td>
<td>6.21</td>
<td>0.00</td>
</tr>
<tr>
<td>Non U.S. Equity</td>
<td>57.24</td>
<td>93.79</td>
<td>100.00</td>
</tr>
<tr>
<td>U.S. Fixed Income</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
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<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
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<tr>
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<td>0.00</td>
<td>0.00</td>
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<td>0.00</td>
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<td>0.00</td>
<td>0.00</td>
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<tr>
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<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>Natural Resources</td>
<td>11.42</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>Cash</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>Probability of ruin ($\pi$)</td>
<td>12.18</td>
<td>29.26</td>
<td>45.86</td>
</tr>
<tr>
<td>Expected Return ($\mu_p$)</td>
<td>4.35</td>
<td>4.74</td>
<td>4.79</td>
</tr>
<tr>
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</table>
SF Portfolio: Numerical Results

The diagram illustrates the Efficient Frontier and various portfolios, including Mean-Variance Portfolio, SF Portfolio, and Asset Class. The x-axis represents Risk (Standard Deviation) while the y-axis represents Expected Return. Different asset classes such as Hedge Funds, Private Equity, U.S. Fixed Income, Private Real Estate, Venture Capital, and Natural Resources are plotted on the graph, showing their positions along the risk-return spectrum.
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Long Term Sustainability

Create 1000 scenarios for the evolution of the endowment fund over 100 years ($t = 0, \ldots, 100$).

- Find returns $R(t)$ using bootstrapped returns $r(t)$
  \[ R(t) = w^T r(t) \]

- Compute the portfolio value $V(t)$ and spending $S(t)$
  \[ V(t) = V(t - 1) e^{R(t)} - S(t) \]
  \[ S(t) = \min\{ V(t), sV(0) \} \]

- Check for bankruptcy:
  \[ V(t) \leq 0 \]
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  \]
Frequency of ruin (%) based on 1000 bootstrap simulations, MLE.

<table>
<thead>
<tr>
<th>Duration (Years)</th>
<th>$\pi = 1%$</th>
<th>$\pi = 5%$</th>
<th>$\pi = 10%$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\leq 10$</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>20</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
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<td>0.3</td>
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<td>4.2</td>
<td>17.4</td>
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<td>11.4</td>
<td>31.0</td>
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<tr>
<td>100</td>
<td>0.0</td>
<td>47.2</td>
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Frequency of ruin (%) based on 1000 bootstrap simulations, MLE.

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<th>Duration (Years)</th>
<th>s=2</th>
<th>s=3</th>
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<td>0.1</td>
<td>1.0</td>
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<tr>
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<td>0.2</td>
<td>19.4</td>
<td>40.9</td>
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<tr>
<td>30</td>
<td>7.5</td>
<td>50.8</td>
<td>72.1</td>
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<tr>
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<td>23.4</td>
<td>70.1</td>
<td>86.2</td>
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<tr>
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<td>39.3</td>
<td>81.9</td>
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<tr>
<td>100</td>
<td>78.4</td>
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<tr>
<td>∞ (model)</td>
<td>12.18</td>
<td>29.26</td>
<td>45.86</td>
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Outline

1. Objective
2. Theoretical Background
3. Statement of Problem
4. Numerical Analysis and Results
5. Long-Term Fund Evolution
6. Conclusions
Conclusions

- A simple, yet insightful asset allocation framework has been proposed for endowment fund management. It combines principles of portfolio asset allocation theory with the probabilistic behavior of perpetual spending.

- SSR and SF portfolios are optimized by concentrated allocations.

- SSR and SF portfolios are found to be on the efficient frontier.

- Optimal probabilities of ruin have been found conservative with respect to simulation outcomes.

- Bootstrap analysis suggests that long term capital preservation and endowment sustainability are not maintained for typical spending rates.